2005

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Advances in Differential Equations Volume 10, Number 11 (2005), 1261–1300

WEAK SOLUTIONS TO THE CAUCHY PROBLEM OF A SEMILINEAR WAVE EQUATION WITH DAMPING AND SOURCE TERMS

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(Submitted by: Viorel Barbu)

Abstract. In this paper we prove local existence of weak solutions for a semilinear wave equation with power-like source and dissipative terms on the entire space $\mathbb{R}^n$. The main theorem gives an alternative proof of the local in time existence result due to J. Serrin, G. Todorova and E. Vitillaro, and also some extension to their work. In particular, our method shows that sources that are not locally Lipschitz in L^2 can be controlled without any damping at all. If the semilinearity involving the displacement has a "good" sign, we obtain global existence of solutions.

1. Introduction

In this paper we study the following semilinear wave equation:

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \text{ a.e. in } \mathbb{R}^n \times [0,\infty); \\ u\big|_{t=0} = u_0; \qquad u_t\big|_{t=0} = u_1. \end{cases} \tag{SW}$$

We will refer to the f nonlinearity as the source term, while g will be called the dissipative term. The initial data u_0 and u_1 are given.

In order to facilitate the presentation, we next list the hypotheses that govern our results.

Assumptions. Suppose that the nonlinearities f and g satisfy the following:

(A0) f is measurable in x, differentiable in t almost everywhere, differentiable in u almost everywhere, and there exists a continuous function k such that for almost every x, t

$$|f_u(x,t,u)| \le k(r) \text{ for a.e. } |u| \le r;$$

(A1) Growth conditions on the source term f:

(i) $f(x,t,0) = 0$;

(ii) $|f(x,t,u)| \le m_1|u|^p + m_2|u|^q$ such that $1 < q < p < 2^*-1$, $m_1, m_2 > 0$, where $2^* = \frac{2n}{n-2}$;

Accepted for publication: May 2005.
AMS Subject Classifications: 35L05, 35L15, 35L70; 35L20.

(A2) $|f_t(x,t,u)| \leq K$ for some $K > 0$;

(A2)* f does not depend on t and $F(x,u) = \int_0^u f(x,v)dv \geq 0$;

(A3) $g = g(x,t,v)$ is measurable in t, differentiable in x, and continuous in v;

(A4) for every x, t the function $v \to g(x,t,v)$ is increasing and $g(x,t,0) = 0$;

(A5) $vg(x,t,v) \geq C_1|v|^{m+1}$ and $|g(x,t,v)| \leq C_2|v|^m$ for some $m \geq 0$;

(A6) $|\nabla_x g(x,t,v)| \leq C|v|$;

(A7) $|g_t(x,t,v)| \leq C|v|$;

(A8) either (a) $1 < p < \frac{2^*}{2}$, $m > 0$, (b) $p + \frac{p}{m} < 2^*$, $m > 0$, or (c) $1 < p < 2^* - 1$, $m \in \{0,1\}$, where p and m are given by (A1), respectively, (A5) above.

In the above assumptions, C, C_1, C_2 represent nonnegative constants which may change from line to line.

The model equation with nonlinearities that satisfy the assumptions (A0)-(A8) is:

$$u_{tt} - \Delta u \pm u|u|^{p-1} \pm u|u|^{q-1} + u_t|u_t|^{m-1} = 0,$$

where $1 < q < p < 2^* - 1$ and $m \geq 0$. The "-" sign corresponds to assumption (iii) (a), whereas the "+" sign gives us a model that satisfies (iii)(b).

The following definition gives a precise description of the type of solutions which are studied in this paper.

Definition 1.1. *Let $\Omega_T := \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is an open connected set with smooth boundary $\partial\Omega$. Suppose the functions f and g satisfy the assumptions* (A1) *and* (A5)*, and further suppose that $u_0 \in H_0^1(\Omega) \cap L^{p+1}(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^{m+1}(\Omega)$.*

A weak solution on Ω_T of the boundary-value problem

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \ \text{in } \Omega \times (0,T); \\ (u,u_t)|_{t=0} = (u_0,u_1); \\ u = 0 \ \text{on } \partial\Omega \times (0,T). \end{cases} \tag{SWB}$$

is any function u satisfying

$$u \in C(0,T; H_0^1(\Omega)) \cap L^{p+1}(\Omega_T), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T),$$

and

$$\int_{\Omega_T} \Big(u(x,s)\phi_{tt}(x,s) + \nabla u(x,s) \cdot \nabla\phi(x,s) + f(x,s,u)\phi(x,s) \\ + g(x,s,u_t)\phi(x,s) \Big) dxds = \int_\Omega \Big(u_1(x)\phi(x,0) - u_0(x)\phi_t(x,0) \Big) dx$$

for every $\phi \in C_c^\infty(\Omega \times (-\infty, T))$.

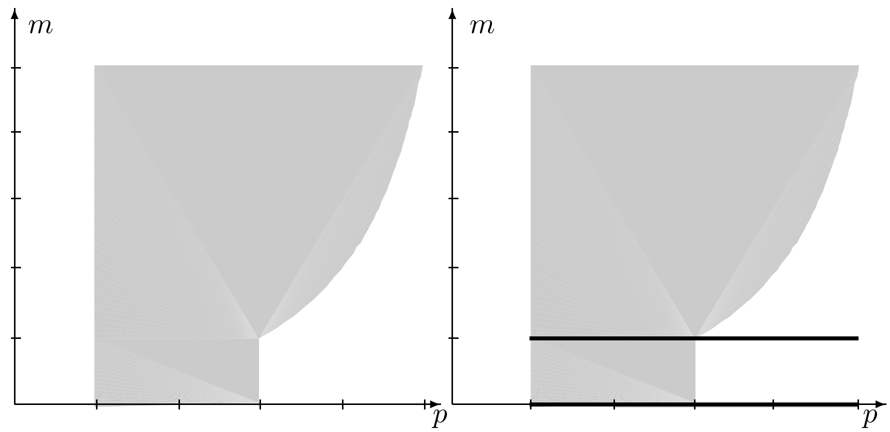
Remark. The above definition remains the same for the Cauchy problem (SW); take $\Omega = \mathbb{R}^n$ with no boundary conditions.

The literature on semilinear wave equations is vast, yet we have complete existence results for only some special cases of semilinearities. First, the existence of weak solutions for the equation with either a power of u or a power of u_t has been studied (see [5, 6, 8, 11]). Even this case presented numerous challenges, and it has been shown that the equation can exhibit blow-up phenomenon when the semilinearity has a "bad" sign. One of the first works on the subject, where the interaction between a power of u and a power of u_t is treated, is the landmark paper authored by J.-L. Lions and W. Strauss [12]. In a series of papers [7, 17, 21, 22], V. Georgiev, J. Serrin, G. Todorova, and E. Vitillaro have brought significant contributions to the study of the wave equation with damping and source terms. Roughly speaking, they prove, for some ranges of exponents, that if the exponent of the source term is higher than the exponent of the damping, one expects blow up of solutions; otherwise, the solutions exist globally. More recently, other semilinearities have been considered; among them, the problem with a degenerate damping term $|u|^k|u_t|^{m-1}u_t$ [2, 14].

The work presented here provides an alternative proof, based on energy methods, to the local existence result by J. Serrin, G. Todorova and E. Vitillaro in [17], and we also obtain a slight improvement for the range of exponents p and m. More precisely, we can show local existence for the Cauchy problem (SW), even when there is no damping (i.e. $m = 0$) for all $1 < p < 2^* - 1$ ($2^* = \frac{2n}{n-2}$), since our proof does not make use of the smoothing effect of the damping. The restriction $p + \frac{p}{m} < 2^*$ for $\frac{2^*}{2} < p < 2^* - 1$ found in [17] appears in our work as well, but we are able to allow p to go all the way up to $2^* - 1$ in the case of Lipschitz damping ($m = 1$).

The result that we obtain holds for finite energy initial data, not necessarily with compact support as is assumed in [17]. The assumption (A7) above is more restrictive than (Q4) (page 8 in [17]), but our work has the advantage of allowing some dependence on t for f, and of having no assumptions on g_v for $m \geq 2$ ((Q5)-(Q6) page 8 in [17]). Regarding global existence, the arguments used in this paper yield global existence results in the special situation when f does not depend on time and $F(u) = \int_0^u f(x,v)dv \geq 0$ (assumption (A2)* - in this case f does not behave as a true source term for the equation).

Some of the tools that we use are the potential well method which goes back to L. E. Payne and D. H. Sattinger in [13, 15], and an idea used by



M. Crandall and L. Tartar in [19] which allows us to solve the problem on the entire space $\mathbb{R}^n$ with arbitrarily large initial data.

In the sequel, we denote by $B(x, R)$ the open ball centered at x, of radius R, and by $B(R)$ the open ball centered at the origin of radius R. $|\Omega|$ is the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^n$. For the norms in the Lebesgue and Sobolev spaces we will use the following notation:

$|\cdot|_{q,\Omega}$ is the norm in $L^q(\Omega)$ and $|\cdot|_q$ is the norm in $L^q(\mathbb{R}^n)$;
$||\cdot||_{k,\Omega}$ is the norm in $H^k(\Omega)$ and $||\cdot||_k$ is the norm in $H^k(\mathbb{R}^n)$;
$||\cdot||_{H_0^1(\Omega)}$ is denoted by $||\cdot||_\Omega$.

The paper is organized as follows. In Section 2 we focus on the boundary-value problem (SWB) with Lipschitz source terms and general damping, for which existence and uniqueness results are known. Section 3 is devoted to obtaining local existence of weak solutions for the Cauchy problem with source terms and damping that satisfy the assumptions (A0)-(A8). We conclude with some remarks and generalizations which can be obtained by using the arguments of this paper.

2. Preliminary results on a bounded domain

This section contains some of the results that we will use in order to prove our main theorem. We record here two existence theorems, that are available in the literature, which deal with a simplified case of the problem (SWB), when the source term $f(x, t, u)$ is a globally Lipschitz function in the u argument, and the dissipative term $g(x, t, u_t)$ is a monotone function in u_t. The first theorem yields existence of strong solutions, while the second deals with weak solutions (see [1], [3], [12] and the Appendix for more discussion of these results). For these solutions we prove finite speed of propagation which will play an important role in the next section when we present our main theorem. We begin by stating the theorem regarding strong solutions (see the Appendix for a proof):

Theorem 2.1. (Existence and uniqueness of strong solutions for dissipative wave equations with Lipschitz source terms) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and the functions $f(x,t,u)$ and $g(x,t,v)$ be under the assumptions* (A0),(A2),(A3)-(A7), *and additionally:*

$$|f(x,t,u) - f(x,t,v)| \leq L|u - v|,$$

for almost every $x \in \mathbb{R}^n$ and for all $t, u, v \in \mathbb{R}$. Let $u_0, u_1 \in H_0^1(\Omega)$ with $u_0 \in H^2(\Omega)$, $G(x,0,u_1) \in L^1(\Omega)$, where G is defined by the formula $G(x,t,v) = \int_0^v g(x,t,y)dy$. Then the initial boundary-value problem:

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \text{ in } \Omega \times (0,T); \\ (u,u_t)|_{t=0} = (u_0,u_1); \\ u = 0 \text{ on } \partial\Omega \times (0,T). \end{cases} \tag{SWB}$$

admits a unique solution u on the time interval $[0,T]$ in the sense of Definition 1.1*; i.e., $u \in C(0,T;H_0^1(\Omega)) \cap L^{p+1}(\Omega_T)$, $u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T)$, with the additional regularity*

$$u \in C^1([0,T];L^2(\Omega)) \quad \text{with} \quad u_{tt},\, \Delta u \in L^2(0,T;L^2(\Omega)).$$

Remark. A solution with the above additional regularity is usually called a strong solution.

A classical technique that we will use is the approximation of the initial data with smooth functions and then passing to the limit in the sequence of approximate solutions. The following theorem will justify this argument.

Theorem 2.2. (Convergence of a sequence of smooth solutions) *Under the assumptions of Theorem* 2.1, *if $(u_{0_\eta}, u_{1_\eta})_{\eta\geq 1}$ is a sequence of smooth functions such that $(u_{0_\eta}, u_{1_\eta}) \to (u_0,u_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, then the solutions u_η provided by Theorem* 2.3 *with initial data (u_{0_η}, u_{1_η}) satisfy*

$$u_\eta(t) \to u(t) \text{ in } H_0^1(\Omega), \quad u_{\eta t}(t) \to u_t(t) \text{ in } L^2(\Omega)$$

for every $t > 0$, where u is the solution of (SWB) with initial data (u_0,u_1).

Proof. We use the same techniques that are used in the proof of Theorem 2.1; i.e. multiply the equation by $u_{\eta t} - u_t$, integrate with respect to the space and then the time variables, and use the Lipschitz assumption for f and the monotonicity for g. Doing so yields:

$$\begin{aligned}
&\int_\Omega |u_{\eta t}(x,t) - u_t(x,t)|^2 + |\nabla u_\eta(x,t) - \nabla u(x,t)|^2 dx \\
&\quad \le \int_\Omega \left[(u_{1_\eta}(x) - u_1(x))^2 + |\nabla u_{0_\eta}(x) - \nabla u_0(x)|^2 \right] dx \\
&\quad + \int_0^t \int_\Omega L \left(|(u_\eta(x,s) - u(x,s))_t|^2 + |\nabla u_\eta(x,s) - \nabla u(x,s)|^2 \right) dxds.
\end{aligned}$$

Poincaré's inequality followed by Gronwall's inequality will show the claimed convergences. In order to finish the proof, we need to show that the limit function u is a solution of (SWB). To this end we invoke the celebrated monotonicity argument due to Lions and Strauss. (A detailed discussion of this can be found in Section 3, page 18 of this paper.) $\square$

The next theorem is basically the statement of Theorem 2.1 under weaker assumptions (less differentiability) for initial data.

Theorem 2.3. (Existence and uniqueness of solutions for dissipative wave equations with Lipschitz source terms) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and the functions $f(x,t,u)$ and $g(x,t,v)$ satisfy assumptions* (A0), (A2), (A3)-(A7), *and f is globally Lipschitz in the last argument with Lipschitz constant L. Let $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ with $G(x,0,u_1) \in L^1(\Omega)$, where G is defined by the formula*

$$G(x,t,v) = \int_0^v g(x,t,y)dy.$$

Then (SWB) admits a unique solution u on the time interval [0, T] in the sense of the Definition 1.1; *i.e.,*

$$u \in C(0,T; H_0^1(\Omega)) \cap L^{p+1}(\Omega_T), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T).$$

Proof. For the pair of initial data $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$, we take a sequence of approximations $u_0^\varepsilon, u_1^\varepsilon \in C_c^\infty(\Omega)$. The regularized sequence satisfies the hypotheses of Theorem 2.1, so we obtain a sequence of smooth solutions u^ε. From Theorem 2.2 we have the existence of a solution u as the limit of u^ε. The uniqueness follows the same way as in Theorem 2.1. $\square$

Two crucial ingredients for the proof of our main existence theorem are a finite speed of propagation result and an energy identity, which are stated and proved below. The energy identity is well known, but the novelty of the finite speed of propagation result consists in the fact that we obtained it for Lipschitz source terms of *arbitrary* sign and general damping which can even be a non-differentiable function.

Theorem 2.4. (Finite speed of propagation) *Consider the problem (SWB) under the hypothesis of Theorem* 2.3. *Then*

(1) *if the initial data* u_0, u_1 *is compactly supported inside the ball* $B(x_0, R) \subset \Omega$, *then* $u(x,t) = 0$ *outside* $B(x_0,\ R+t)$*;*

(2) *if* $(u_0, u_1), (v_0, v_1)$ *are two pairs of initial data with compact support, with the corresponding solutions* $u(x,t)$, *respectively* $v(x,t)$, *and* $u_0(x) = v_0(x)$ *for* $x \in B(x_0,R) \subset \Omega$, *then* $u(x,t) = v(x,t)$ *inside* $B(x_0,\ R-t)$ *for any* $t < R$.

Proof. Part (1) The proof presented here extends an argument used for the linear wave equation by L. Tartar [20].

Assume for now that $f(x,t,u) = 0$ for $|x - x_0| \geq R + t$. Since the equation is invariant under translations, without loss of generality we can take $x_0 = 0$. First we approximate the initial data uniformly by smooth functions (u_{0_η}, u_{1_η}) with compact support inside $B(R_\eta)$, with $R_\eta \nearrow R$ as $\eta \to 0$. By Theorem 2.1, for any $T > 0$, the solution of:

$$\begin{cases} u_{\eta tt} - \Delta u_\eta + f(x,t,u_\eta) + g(x,t,u_{\eta t}) = 0 \text{ in } \Omega \times (0,T); \\ (u_\eta, u_{\eta t})|_{t=0} = (u_{\eta 0}, u_{\eta 1}); \\ u_\eta = 0 \text{ on } \partial\Omega \times (0,T) \end{cases} \tag{SWB$_\eta$}$$

exists on $[0,T]$ and it has the regularity of a strong solution.

Consider a function ϕ_η with $\phi_\eta(r) = 0$ on $(-\infty, R_\eta]$, $\phi_\eta(r) > 0$ on (R_η, ∞), such that $\phi'(r) \geq 0$ on $\mathbb{R}$. Since $u_{\eta t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, we are allowed to multiply (SW$_\eta$) by $u_{\eta t}(t,x)\phi_\eta(|x| - t)$, for any $0 < t < T$. The quantity:

$$I_\eta(t) := \int_{\mathbb{R}^n} (|u_{\eta t}(x,t)|^2 + |\nabla u_\eta(x,t)|^2)\phi_\eta(|x| - t)\, dx$$

is well defined and assume for now that $\frac{dI_\eta}{dt} \leq 0$. It can be easily seen that $I_\eta(0) = 0$, since

$$I_\eta(0) = \int_{|x|<R_\eta} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\, dx \\ + \int_{|x|>R_\eta} (|u_{1_\eta}(x)|^2 + |\nabla u_{0_\eta}(x)|^2)\phi_\eta(|x|)\, dx.$$

The first integral is 0 since $\phi_\eta(|x|) = 0$ for $|x| < R_\eta$. The initial data has support inside the domain $|x| < R_\eta$, so the second integral is zero.

The assumption that the mapping $t \to I_\eta(t)$ is decreasing leads us to $I_\eta(t) \leq I_\eta(0) = 0$, which means that $u_\eta(x,t) = 0$ if $|x| - t > R_\eta$. We pass to the limit in η (see Theorem 2.2) to obtain $u(x,t) = 0$ for $|x| - t > R$, and this concludes the proof.

It suffices then to prove that $\frac{dI_\eta}{dt} \leq 0$. For the regularized initial data we have $u_\eta \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta t} \in L^\infty(0,T; H_0^1(B(R_\eta)))$, $u_{\eta tt} \in L^1(0,T; L^2(B(R_\eta)))$, which enables us to compute (we drop the subscript η in the remainder of the proof)

$$\begin{aligned}
\frac{dI}{dt}(t) &= \int_{\mathbb{R}^n} 2\phi(|x|-t)(u_t u_{tt} + \sum_{i=1}^n u_{x_i} u_{tx_i})(x,t)\,dx \\
&- \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx = \int_{\mathbb{R}^n} 2\phi(|x|-t)(u_t u_{tt})(x,t)\,dx \\
&- \int_{\mathbb{R}^n} \sum_{i=1}^n \left((2\phi(|x|-t)u_{x_i})_{x_i} u_t\right)(x,t)\,dx - \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx \\
&= \int_{\mathbb{R}^n} 2\phi(|x|-t)((u_{tt}-\Delta u)u_t)(x,t)\,dx - \int_{\mathbb{R}^n} \sum_{i=1}^n 2\phi'(|x|-t)\frac{x_i}{r}(u_{x_i} u_t)(x,t)\,dx \\
&\qquad - \int_{\mathbb{R}^n} \phi'(|x|-t)(u_t^2 + |\nabla u|^2)(x,t)\,dx.
\end{aligned}$$

By (SWB)

$$u_{tt} - \Delta u = -f(x,t,u) - g(x,t,u_t),$$

hence the assumptions on the support of ϕ and f, together with the fact that g is nondecreasing, make the first term of the last equality negative. We factor out $\phi'(|x|-t)$ in the other two terms, and since $\phi'(r) \geq 0$ for every r, it is enough to show that

$$u_t^2 + |\nabla u|^2 + 2\sum_{i=1}^n \frac{x_i}{|x|} u_{x_i} u_t \geq 0, \tag{2.1}$$

which is obtained by summing the inequalities:

$$\left(\frac{x_i}{|x|} u_t + u_{x_i}\right)^2 \geq 0 \quad \text{for all } i = 1, ..., n.$$

It remains to show that the function f vanishes for $|x - x_0| \geq R + t$. A fixed-point argument will establish this fact now. Consider the iterative equation:

$$\begin{cases} u_{tt}^{k+1} - \Delta u^{k+1} + f(x,t,u^k) + g(x,t,u_t^{k+1}) = 0 \\ (u^{k+1}, u_t^{k+1})|_{t=0} = (u_0, u_1) \\ u^{k+1} = 0 \text{ on } \partial\Omega \times (0,T), \end{cases}$$

for every $k \in \mathbb{N}$, with $(u^0, u_t^0) = (u_0, u_1)$. The existence of a unique weak solution is guaranteed by Theorem 2.3. An induction argument, together with the first part of the proof, will show that $u^k(x,t) = 0$ for $|x - x_0| > R + t$, for every $k \in \mathbb{N}$. It is enough then to show that $u^k(x,t) \to u(x,t)$ almost everywhere as $k \to \infty$. Since f is Lipschitz we obtain that $f(x,t,u^k(x,t))$, which is zero for $|x - x_0| \geq R + t$, converges almost everywhere to $f(x,t,u(x,t))$, hence f vanishes outside the cone $|x - x_0| < R + t$. The sequence of difference functions $v^k(x,t) := u^k(x,t) - u(x,t)$ satisfies:

$$\begin{cases} v_{tt}^{k+1} - \Delta v^{k+1} + f(x,t,v^k + u) - f(x,t,u) \\ \qquad\qquad +g(x,t,v_t^{k+1} + u_t) - g(x,t,u_t) = 0 \\ (v^{k+1}, v_t^{k+1})|_{t=0} = (0,0) \\ v^{k+1} = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Upon multiplication by v_t^{k+1} and integration over $(0,t) \times \mathbb{R}^n$, we use the monotonicity of g to derive the following inequality:

$$\begin{aligned} &\int_{\mathbb{R}^n} (v_t^{k+1}(x,t))^2 + |\nabla v^{k+1}(x,t)|^2 \, dx \\ &\leq \int_0^t \int_{\mathbb{R}^n} 2|f(x,t,v^k + u) - f(x,t,u)||v_t^{k+1}(x,s)| dx\, ds, \end{aligned}$$

which by the Lipschitz assumption on f is

$$\leq \int_0^t 2L|v^k(s)|_2 |v_t^{k+1}(s)|_2 \, ds \leq L \int_0^t |v^k(s)|_2^2 + |v_t^{k+1}(s)|_2^2 \, ds.$$

We now need a bound for $\displaystyle\int_0^t |v^k(s)|_2^2 ds$, which we obtain by writing:

$$\begin{aligned} |v^k(t)|_2^2 &= 2\int_0^t \int_{\mathbb{R}^n} v^k(x,s) v_t^k(x,s) dx\, ds \\ &\leq \int_0^t \int_{\mathbb{R}^n} (v^k(x,s))^2 + (v_t^k(x,s))^2 dx\, ds. \end{aligned}$$

Gronwall's inequality for the function $|v^k(s)|_2^2$ will give us for any $t < T$ the bound:

$$|v^k(t)|_2^2 \leq e^T \int_0^t |v_t^k(s)|_2^2 ds.$$

At this point, to simplify the writing let

$$\phi^k(t) := \int_0^t |v_t^k(s)|_2^2 + |\nabla v^k(t)|_2^2 ds.$$

By summarizing the estimates above, we have that ϕ^{k+1} satisfies the inequality:

$$\phi_t^{k+1}(t) \leq L\phi^{k+1}(t) + C\phi^k(t),$$

which after integration becomes:

$$\phi^{k+1}(t) \leq C\int_0^t e^{L(t-s)}\phi^k(s)ds \leq Ce^{LT}\int_0^t \phi^k(s)ds.$$

A simple induction argument will show that:

$$\phi^{k+1}(t) \leq K\frac{C^{k+1}e^{LT(k+1)}t^{k+1}}{(k+1)!},$$

where K is a bound on $|\phi^1(t)|$ for all t in $[0,T]$. Thus we proved the convergence for $u^k(x,t)$ almost everywhere (x,t), so $u(x,t) = 0$ outside the domain of dependence, i.e. for $|x - x_0| \geq R + t$. Recall that this actually is proven for the sequence of approximated solutions u_η. By Theorem 2.2 we have the convergence $u_\eta(t) \to u(t) \in H_0^1(\Omega)$, hence $u_\eta(t) \to u(t)$ almost everywhere. This concludes the proof of Part (1).

Part (2). We follow here a similar argument as in Part (1). Initially, we work under the assumption that

$$f(x,t,u(x,t)) = f(x,t,v(x,t)) \tag{2.2}$$

for $|x| < R - t$ (again, take $x_0 = 0$). The difference $u - v$ satisfies:

$$\begin{cases} (u-v)_{tt} - \Delta(u-v) + f(x,t,u) - f(x,t,v) + g(x,t,u_t) - g(x,t,v_t) = 0; \\ ((u-v),(u-v)_t)\,|_{t=0} = (0,0) \\ u - v = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Consider a function ψ strictly positive on $(-\infty, R)$, such that $\psi(r) = 0$ on $[R,\infty)$ and $\psi'(r) \leq 0$ everywhere. Define the function $J(t)$ by

$$J(t) := \int_{\mathbb{R}^n} \Big((u_t(x,t) - v_t(x,t))^2 + |\nabla(u(x,t) - v(x,t))|^2\Big)\psi(|x| - t)dx.$$

We will show that $\dfrac{dJ}{dt} \leq 0$. As before, this will show that $u(x,t) = v(x,t)$ on the support of $\psi(|x| + t)$, i.e. if $|x| < R - t$. The proof here is similar to that in (1):

$$\begin{aligned} \frac{dJ}{dt} = \int_{\mathbb{R}^n} & 2\psi(|x|+t)[((u-v)_{tt} - \Delta(u-v))(u-v)_t](x,t)\,dx \\ & + \int_{\mathbb{R}^n}\sum_{i=1}^n 2\psi'(|x|+t)\frac{x_i}{r}[(u-v)_{x_i}(u-v)_t](x,t)\,dx \end{aligned}$$

$$+ \int_{\mathbb{R}^n} \psi'(|x|+t)[(u-v)_t^2 + |\nabla(u-v)|^2](x,t)\,dx.$$

We use the equality

$$(u-v)_{tt} - \Delta(u-v) = -f(x,t,u) + f(x,t,v) - g(x,t,u_t) + g(x,t,v_t),$$

(2.1), (2.2), and the assumptions on the support of ψ to obtain that J is decreasing. In order to prove (2.2), we use the following iterative schemes with $u^1 = u_0$ and $v^1 = v_0$:

$$\begin{cases} u_{tt}^{k+1} - \Delta u^{k+1} + f(x,t,u^k) + g(x,t,u_t^{k+1}) = 0 \\ (u^{k+1}, u_t^{k+1})|_{t=0} = (u_0, u_1) \\ u^{k+1} = 0 \text{ on } \partial\Omega \times (0,T), \end{cases}$$

and

$$\begin{cases} v_{tt}^{k+1} - \Delta v^{k+1} + f(x,t,v^k) + g(x,t,v_t^{k+1}) = 0 \\ (v^{k+1}, v_t^{k+1})|_{t=0} = (v_0, v_1) \\ v^{k+1} = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

Again, the existence and uniqueness of the solutions u^{k+1} and v^{k+1} is guaranteed by Theorem 2.1. By the first part of the proof, since $(u_0, u_1) = (v_0, v_1)$ we have that $u^k = v^k$, which implies that $u^{k+1} = v^{k+1}$ on the desired domain. Since we also have that $u^1 = v^1$, by the induction principle, the equality $u^k = v^k$ holds true for all k's. The argument will be complete after showing $u^k - v^k \to 0$ almost everywhere as $k \to \infty$. We subtract the above iterative schemes and obtain:

$$\begin{cases} (u^{k+1} - v^{k+1})_{tt} - \Delta(u^{k+1} - v^{k+1}) + f(x,t,u^k) - f(x,t,v^k) \\ \qquad\qquad\qquad\qquad + g(x,t,u_t^{k+1}) - g(x,t,v_t^{k+1}) = 0 \\ (u^{k+1} - v^{k+1}, (u^{k+1} - v^{k+1})_t)|_{t=0} = (0,0) \\ u^{k+1} - v^{k+1} = 0 \text{ on } \partial\Omega \times (0,T). \end{cases}$$

An argument identical to the one used in (1) for the sequence v^k concludes the proof. $\square$

The proof of the following proposition is nontrivial, but it can be obtained by a modification of the argument in Lemma 8.3 in [10]. Note that for **strong** solutions, the proof is immediate.

Proposition 2.5. (The energy identity) *If u is a weak solution of (SWB), under the assumptions of Theorem 2.3 we have the following equality:*

$$E(t) + \int_\Omega F(x,t,u(x,t))\,dx - \int_0^t \int_\Omega f_t(x,s,u(x,s))\,dx\,ds$$

$$+\int_0^t \int_\Omega g(x,s,u_t(x,s))u_t(x,s)\,dx\,ds = E(0), \quad (2.3)$$

where $E(t) := \frac{1}{2}|u_t(t)|^2_{2,\Omega} + \frac{1}{2}|\nabla u(t)|^2_{2,\Omega}$.

3. THE CAUCHY PROBLEM

In this section we state and prove the main result of this paper. In comparison with the existence theorem from the previous section, we remark that here the bounded set Ω is replaced by $\mathbb{R}^n$, and the source term f is not required to be Lipschitz.

Theorem 3.1. (Existence of weak solutions) *Let* $(u_0, u_1) \in H^1_0(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *and consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + f(x,t,u) + g(x,t,u_t) = 0 \ \text{a.e. in } \mathbb{R}^n \times [0,\infty); \\ u|_{t=0} = u_0; \quad u_t|_{t=0} = u_1. \end{cases} \quad \text{(SW)}$$

Assume $G(x,0,u_1) \in L^1(\mathbb{R}^n)$, *where* $G(x,t,v) = \int_0^v g(x,t,u)du$, *and the validity of assumptions (A0)-(A8). Then, there exists a time* $0 < T < 1$ *such that (SW) admits a weak solution on* $[0,T]$ *in the sense of Definition* 1.1. *In addition, if (A2)* is satisfied, then the solution is global, so* T *can be taken arbitrarily.*

Remark. The bound $T < 1$ is artificially imposed; the true restriction for the time of existence is due to the interaction of the semilinearities f and g and it will be presented in more detail in the proof.

Proof. With or without assumption (A2)*, the proof follows the same argument, so we will make the necessary adjustments when needed and show how the assumption (A2)* gives a stronger result.

We start by assuming that f is globally Lipschitz in the last argument; i.e., there exists a constant $L > 0$ such that:

$$|f(x,t,u) - f(x,t,v)| \le L|u - v|. \quad (3.1)$$

Also, in the beginning we take u_0, u_1 with compact support inside a ball of radius R, so that we deal with the problem on a bounded domain. With such f, g, u_0, u_1 the existence and regularity results of Section 2 are available.

As part of a compactness argument, our goal is to obtain bounds for $|\nabla u(t)|_\Omega$, where Ω is a bounded domain which contains the support of the initial data. This is straightforward from the energy identity if we assume the positivity conditions (A2)* on the antiderivative F. If we work under the assumption (A1)(a), in a first step we additionally impose some "smallness" assumptions in order to prove that $|\nabla u(t)|_\Omega < \alpha$ for any $t < 1$.

Next, we construct Lipschitz approximations f_ε for the general nonlinearity f that satisfies the assumptions (A0)-(A2). We apply the results obtained in the first step (where the bounds will not depend on ε) to the sequence of solutions u_ε and pass to the limit to get a solution of the problem (SW) on a bounded domain. We will eliminate the "smallness" restrictions imposed in the case of (A1)(a), and the fact that the initial data has compact support through a "patching" argument.

Step 1. As mentioned above, we start by assuming that f is Lipschitz, so that (3.1) holds. Fix $x_0 \in \mathbb{R}^n$, and consider the initial data of (SW) supported inside the ball $B(x_0, R)$, with zero boundary conditions on a domain sufficiently large that contains $B(x_0, R)$. Without loss of generality, we can assume that $x_0 = 0$, since the equation is invariant under translations in space.

First, we will discuss the case when f satisfies (A1)(a).

Due to the finite speed of propagation property (Theorem 2.4), we have that $u(x,t)$ is zero outside $B(R+t)$; hence, for $t < 1$, $u(x,t)$ is supported inside the set $\Omega := B(R+1)$. In this case we impose the following "smallness" conditions on the initial data:

$$|\nabla u_0|_\Omega < \alpha, \quad \frac{1}{2}|u_1|^2_\Omega + \frac{1}{2}|\nabla u_0|^2_\Omega + \int_\Omega F(x, 0, u_0(x))\, dx + K|\Omega| < \Phi(\alpha), \tag{3.2}$$

where α and Φ will be chosen later, K is the constant in (A2), and $\Omega = B(R+1)$.

For $p < 2^*$ and any $v \in H^1_0(\Omega)$ we will need the following inequality:

$$|v|_{p,\Omega} \le C(R+1)^{n\frac{2^*-p}{2^*p}} |\nabla v|_\Omega. \tag{3.3}$$

This is a consequence of the Hölder inequality:

$$|v|_{p,\Omega} \le \Big(\int_\Omega |v|^{\frac{2^*}{p}p} dx \Big)^{\frac{1}{2^*}} \Big(\int_\Omega dx \Big)^{\frac{2^*-p}{2^*p}} \le |v|_{2^*} \big(\frac{\omega_n}{n}(R+1)^n\big)^{\frac{2^*-p}{2^*p}},$$

and of the Sobolev imbedding theorem:

$$|v|_{2^*} \le C^* |\nabla v|_\Omega,$$

where C^* depends only on n, and in (3.3) we take $C = C^*(\frac{\omega_n}{n})^{\frac{2^*-p}{2^*p}}$, where $\frac{\omega_n}{n}$ is the volume of the unit sphere in $\mathbb{R}^n$.

Since $u(t) \in H^1_0(B(R+t))$ implies $u(t) \in H^1_0(\Omega)$ for $t < 1$, the inequality (3.3) will hold for any $u(t)$ with $0 < t < 1$.

By (A4) and the energy identity we obtain:

$$\frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx$$
$$\leq K|\Omega| + \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F(x,0,u_0(x))\,dx = K|\Omega| + E(0). \quad (3.4)$$

The growth assumption for F given in (A1)(iii)(a), followed by an application of the inequality (3.3) yields:

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx \geq \frac{1}{2}|\nabla u(t)|_\Omega^2 - m_1|u(t)|_{p,\Omega}^p - m_2|u(t)|_{q,\Omega}^q$$
$$\geq \frac{1}{2}|\nabla u(t)|_\Omega^2 - m_1C^p|\nabla u(t)|_\Omega^p(R+1)^{n\frac{2^*-p}{2^*}}$$
$$- m_2C^q|\nabla u(t)|_\Omega^q(R+1)^{n\frac{2^*-q}{2^*}}. \quad (3.5)$$

The right-hand side of the above inequality will be analyzed with the aid of the function:

$$\Phi(x) = \frac{x^2}{2} - Ax^p - Bx^q, \quad x \geq 0. \quad (3.6)$$

where $A, B > 0$.

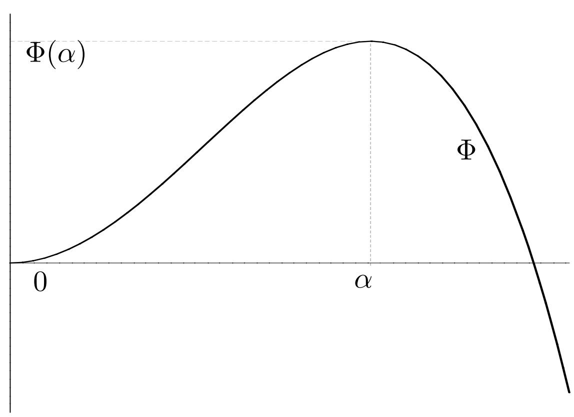


FIGURE 1. Graph of Φ

For $p, q > 2$, Φ has exactly 2 critical points on the positive semiaxis: $x = 0$ and $x = \alpha$, where α is the only positive root of the equation $pA\alpha^{p-2} + qB\alpha^{q-2} = 1$. At $x = 0$, Φ has a local minimum and at $x = \alpha$ a global maximum. In (3.6) we take:

$$A = m_1 C^p (R+1)^{n\frac{2^*-p}{2^*}}, \qquad B = m_2 C^q (R+1)^{n\frac{2^*-q}{2^*}}.$$

($R+1$ is related to the restriction $t < 1$.) We point out that the root α depends on R, which measures the size of the support of the initial data.

Assume that $|\nabla u(s)|_\Omega < \alpha$ for all $s \in [0, s_0)$, for some $s_0\,(s_0 > 0$, as $|\nabla u_0|_\Omega < \alpha$ and $t \to |\nabla u(t)|_\Omega$ is continuous). With the new notation, (3.5) becomes:

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx \geq \frac{1}{2}|\nabla u(t)|_\Omega^2 - A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q,$$

which combined with (3.4) gives us:

$$\begin{aligned}
\frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 &- A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q \\
&\leq \frac{1}{2}|u_t(t)|_\Omega^2 + \frac{1}{2}|\nabla u(t)|_\Omega^2 + \int_\Omega F(x,t,u(x,t))\,dx \\
&\leq K|\Omega| + \frac{1}{2}|u_1|_\Omega^2 + \frac{1}{2}|\nabla u_0|_\Omega^2 + \int_\Omega F(x,0,u_0(x))\,dx < \Phi(\alpha),
\end{aligned} \tag{3.7}$$

the last inequality being part of the "smallness" hypothesis which we assumed in Step 1. Therefore

$$\frac{1}{2}|\nabla u(t)|_\Omega^2 - A|\nabla u(t)|_\Omega^p - B|\nabla u(t)|_\Omega^q = \Phi(|\nabla u(t)|_\Omega) < \Phi(\alpha) \tag{3.8}$$

so, by the continuity in time of $|\nabla u(t)|_\Omega$ we get:

$$|\nabla u(t)|_\Omega < \alpha \tag{3.9}$$

for any $t < 1$. Otherwise, $\Phi(|\nabla u(t)|_\Omega) \geq \Phi(\alpha)$ for the times t which do not satisfy (3.9), but this would contradict (3.8). (In other words, if we start in the well of the graph of Φ at time $t = 0$ with $|\nabla u_0|_\Omega < \alpha$, we cannot get out, so $|\nabla u(t)|_\Omega$ remains bounded by α.) We make the remark that the bound $t < 1$ is related to the choice of the domain $\Omega = B(R+1)$.

If f satisfies (A2)*, we easily obtain the bound $|\nabla u(t)|_\Omega < C(u_0, u_1)$ from the energy identity, since we have that $F \geq 0$.

The time t was artificially bounded by 1; the true restriction is a consequence of the arguments above, and it will be explained more in the last step. This will give us only a local result in time under the hypothesis (A1)(a). There will be no constraints if assumptions (A2)* are satisfied, hence we obtain global existence of solutions.

Step 2. In this step we will construct truncations for the initial data, for which the "smallness" assumptions are satisfied. First consider a pair of initial data (u_0, u_1) such that $u_0 \in H^2(\mathbb{R}^n), u_1 \in H^1(\mathbb{R}^n)$, and $G(x, 0, u_1) \in L^1(\mathbb{R}^n)$ (recall that $G(x, t, v) = \int_0^v g(x, t, y)\, dy$). The higher differentiability assumptions on the initial data will be later removed. For now, we keep the Lipschitz assumptions for f.

Fix $x_0 \in \mathbb{R}^n$. We will find a domain Ω around x_0, small enough, and construct a new pair of initial data (u_0^*, u_1^*) such that they satisfy (3.2) inside Ω. We apply the results of Step 1 to obtain bounds for times $t < 1$ for the new solution u^* generated by the initial data (u_0^*, u_1^*).

Again, we first analyze the case when (A1)(a) is satisfied. From now on let α be the critical point of the function Φ from Step 1, with the coefficients A, B corresponding to $R = 1$. Hence, in the sequel α will depend *only* on the norm of the initial data.

We find $\rho < 1$ small enough such that

$$
\begin{aligned}
&\rho < \Big(\frac{\Phi(\alpha)}{4K\omega_n}\Big)^{1/n}, \quad |\nabla u_0|_{B(x_0,\rho)} < \frac{\alpha}{2}, \quad \text{and} \quad |\nabla u_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{8},\\
&2(C^*\omega_n)^{\frac{1}{n}}\left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\right) \le \frac{\alpha}{2}, \text{ and}\\
&4{C^*}^2\omega_n^{\frac{2^*-2}{2^*}}|u_0|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{8}, \quad \frac{1}{2}|u_1|^2_{B(x_0,\rho)} \le \frac{\Phi(\alpha)}{4},\\
&m_1(C^*)^p\Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^p \le \frac{\Phi(\alpha)}{8},\\
&m_2(C^*)^q\Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\big(\frac{2}{\rho}+1\big)\Big)^q \le \frac{\Phi(\alpha)}{8},
\end{aligned}
\tag{3.10}
$$

where C^* is the constant from the Sobolev inequality. To have these conditions satisfied, it is enough to take $\rho < (\frac{\Phi(\alpha)}{4K\omega_n})^{1/n}$ such that

$$
\begin{aligned}
|\nabla u_0|_{B(x_0,\rho)} < \min\Big\{\frac{\alpha}{2}, \sqrt{\frac{\Phi(\alpha)}{8}}, \frac{\alpha}{4(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{2C^*}\Big(\frac{\Phi(\alpha)}{8m_1}\Big)^{1/p},\\
\frac{1}{2C^*}\Big(\frac{\Phi(\alpha)}{8m_2}\Big)^{1/q}\Big\},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
|u_0|_{B(x_0,\rho)} < \min\Big\{\frac{\alpha}{4(C^*\omega_n)^{\frac{1}{n}}}, \frac{1}{\omega_n C^*}\sqrt{\frac{\Phi(\alpha)}{32\omega_n^{\frac{2^*-2}{2^*}}}}, \frac{1}{8C^*}\left(\frac{\Phi(\alpha)}{8m_1}\right)^{1/p},\\
\frac{1}{8C^*}\Big(\frac{\Phi(\alpha)}{8m_2}\Big)^{1/q}\Big\},
\end{aligned}
\tag{3.12}
$$

and

$$|u_1|^2_{B(x_0,\rho)} \le \sqrt{\frac{\Phi(\alpha)}{8}}. \tag{3.13}$$

The existence of such a ρ, independent of $x_0 \in \mathbb{R}^n$, is motivated by the equi-integrability of the functions $u_0, \nabla u_0, u_1$. More precisely, for each of the functions $u_0, \nabla u_0, u_1$ we apply the following result of classical analysis:

If $f \in L^1(A)$, with A a measurable set, then for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f(x)|dx < \varepsilon$, for every measurable set $E \subset A$ of measure less than δ (see [4]). Note that δ does not depend on E, hence ρ does not vary with x_0.

It is possible that $u_0 \notin H_0^1(B(x_0,\rho))$ since it does not necessarily have zero trace on the boundary, so in order to apply the results of Step 1, we multiply u_0 by a cutoff function. For ε sufficiently small (to be chosen later), choose θ_ε, a twice differentiable cutoff function, obtained by smoothing the Lipschitz graph:

$$\theta_{0_\varepsilon}(x) = \begin{cases} 1, & |x-x_0| \le \rho - \varepsilon \\ \dfrac{\rho - |x-x_0|}{\varepsilon}, & \rho - \varepsilon \le |x-x_0| \le \rho \\ 0, & |x-x_0| \ge \rho. \end{cases}$$

Choose an appropriate smoothing operator for θ_{0_ε} such that we have:

$$|\theta_\varepsilon|_{\infty,B(x_0,\rho)} \le 1, \quad |\nabla\theta_\varepsilon|_{\infty,B(x_0,\rho)} \le \frac{1}{\varepsilon}. \tag{3.14}$$

The product $\theta_\varepsilon u_0 =: u_0^*$ belongs to $H_0^1(B(x_0,\rho)) \cap H^2(B(x_0,\rho))$, but our goal is to also have the inequality (3.2) satisfied by the new pair of initial data (u_0^*, u_1^*) (note that u_1 already enjoys all the desired smoothness, so we can take $u_1^* = u_1$). As the computations below will show, the domain over which the integrals in the inequality (3.2) are considered will have to be taken smaller. Mainly, this is due to the fact that the gradient of the new initial data may increase when multiplied by the cutoff function.

We have

$$\nabla u_0^* = \theta_\varepsilon \nabla u_0 + u_0 \nabla\theta_\varepsilon.$$

Hence,

$$|\nabla u_0^*|_{B(x_0,\rho)} \le |\theta_\varepsilon|_{\infty,B(x_0,\rho)} |\nabla u_0|_{B(x_0,\rho)} + |\nabla\theta_\varepsilon|_{\infty,B(x_0,\rho)} |u_0|_{B(x_0,\rho)}.$$

By (3.14) and by Hölder's inequality, the above quantity is:

$$< \frac{\alpha}{2} + |u_0|_{2*,B(x_0,\rho)} |B(x_0,\rho)|^{\frac{1}{n}} \frac{1}{\varepsilon}.$$

Choose $\varepsilon < \rho$, so the above inequality, with the aid of Sobolev's inequality, becomes:

$$|\nabla u_0^*|_{B(x_0,\rho)} \le \frac{\alpha}{2} + (C^*\omega_n)^{\frac{1}{n}} \left(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)} \right) \overset{(3.11),(3.12)}{\le} \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

This proves that for any $\rho^* < \rho$ we have

$$|\nabla u_0^*|_{B(x_0,\rho^*)} < \alpha.$$

We claim that (u_0^*, u_1^*) satisfies the second inequality of (3.2) on the smaller ball of radius ρ^* ($\rho^* < \rho$ will be found later); i.e.

$$\frac{1}{2}|u_1^*|^2_{B(x_0,\rho^*)} + \frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho^*)} + \int_{B(x_0,\rho^*)} F(x,0,u_0^*(x))\,dx \\ + K|B(x_0,\rho^*)| < \Phi(\alpha). \tag{3.15}$$

We prove this inequality by estimating each term. For the first term we have:

$$\frac{1}{2}|u_1^*|^2_{B(x_0,\rho^*)} \le \frac{1}{2}|u_1^*|^2_{B(x_0,\rho)} = \frac{1}{2}|u_1|^2_{B(x_0,\rho)} < \frac{\Phi(\alpha)}{4}.$$

Also,

$$\frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho^*)} \le \frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho)} \le |\theta|^2_{\infty,B(x_0,\rho)}|\nabla u_0|^2_{B(x_0,\rho)} + \\ + |\nabla\theta|^2_{\infty,B(x_0,\rho)}|u_0|^2_{B(x_0,\rho)} < \frac{\Phi(\alpha)}{8} + \frac{1}{\varepsilon^2}|u_0|^2_{B(x_0,\rho)}. \tag{3.16}$$

Choose $\rho^* < \frac{\rho}{2}$ such that

$$\frac{4}{\rho^2}|u_0|^2_{B(x_0,\rho^*)} < \frac{\Phi(\alpha)}{8}. \tag{3.17}$$

Take $\varepsilon := \rho - \rho^*$ (this satisfies the earlier restriction that $\varepsilon < \rho$). Since $\varepsilon > \frac{\rho}{2}$, we have that

$$\frac{1}{\varepsilon^2}|u_0|^2_{B(x_0,\rho^*)} < \frac{\Phi(\alpha)}{8}.$$

Therefore,

$$\frac{1}{2}|\nabla u_0^*|^2_{B(x_0,\rho^*)} < \frac{\Phi(\alpha)}{4}.$$

For the third term, we use the fact that $\rho* < \rho$, the assumption (A1)(a), the Sobolev embedding theorem and the restrictions for ρ in (3.11), (3.12) to obtain:

$$\int_{B(x_0,\rho*)} F(x,0,u_0^*(x))\,dx \le \int_{B(x_0,\rho)} F(x,0,u_0^*(x))\,dx$$

$$
\begin{aligned}
&\leq \int_{B(x_0,\rho)} (m_1|u_0^*(x)|^p + m_2|u_0^*(x)|^q)\, dx \\
&\leq m_1(C^*)^p \left(|\nabla u_0^*|_{B(x_0,\rho)} + |u_0^*|_{B(x_0,\rho)}\right)^p + m_2(C^*)^q \Big(|\nabla u_0^*|_{B(x_0,\rho)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + |u_0^*|_{B(x_0,\rho)}\Big)^q \\
&\leq m_1(C^*)^p \Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\Big(\frac{2}{\rho}+1\Big)\Big)^p \\
&+ m_2(C^*)^q \Big(|\nabla u_0|_{B(x_0,\rho)} + |u_0|_{B(x_0,\rho)}\Big(\frac{2}{\rho}+1\Big)\Big)^q < \frac{\Phi(\alpha)}{8} + \frac{\Phi(\alpha)}{8} = \frac{\Phi(\alpha)}{4}.
\end{aligned}
$$

We also have that $K|B(x_0, \rho*)| < \frac{\Phi(\alpha)}{4}$, so by summing the above inequalities we have (3.15). Next, we approximate initial data $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ by smooth functions and pass to the limit in (3.15) to obtain the conclusions of this step for general initial data.

Step 3. At this time we approximate the source term f by Lipschitz functions f_ε such that f_ε satisfy the assumptions (A0) with a new function k_ε instead of k, and (A1), (A2) with constants $m_1, m_2, p, q, 1+K$. Obviously, the Lipschitz constants for each of the functions f_ε may depend on ε. The exact procedure of approximation will be presented at the end of Step 3. We note that the estimates regarding the functions f_ε that follow from the energy identity will basically remain unaffected (replace K by $1+K$).

In the case when (A1)(a) is assumed, since the growth condition uses the same m_1, m_2, p, q for f_ε as for f, the coefficients A, B from (3.6), corresponding to f_ε, as well as the root α, and the radius ρ, chosen in Step 1, *will not depend on* ε.

With the notation of the previous sections, we solve the problem with the initial data (u_0^*, u_1^*) (where (u_0^*, u_1^*) are obtained in Step 2) and nonlinearities f_ε, g, and zero boundary conditions on a domain large enough that includes $B(x_0, \rho^*)$. We obtain that the solution u^* enjoys the regularity stated in Theorem 2.3, and furthermore, u^* satisfies the estimate:

$$
|\nabla u_\varepsilon^*(t)|_{B(x_0,\rho^*)} < \alpha, \tag{3.18}
$$

for $0 \leq t \leq \rho^*/2$. From the energy identity (2.3), hypothesis (A2), and the fact that g is increasing, the following inequality results:

$$
\begin{aligned}
|u_{\varepsilon_t}^*(t)|^2_{2,B(x_0,\rho^*)} + |\nabla u_\varepsilon^*(t)|^2_{2,B(x_0,\rho^*)} + \int_{B(x_0,\rho^*)} F_\varepsilon(x,t,u_\varepsilon^*(x,t))dx \\
\leq 2K|B(x_0,\rho^*)| + E(0),
\end{aligned}
$$

so, with the growth condition (A1)(a) on F^{ε}, Sobolev's inequality and (3.18), we obtain the bound:

$$\begin{aligned}
&|u^*_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho^*)} + |\nabla u^*_{\varepsilon}(t)|^2_{2,B(x_0,\rho^*)} \qquad (3.19)\\
&\leq 2K|B(x_0,\rho^*)| + E(0) + \int_{B(x_0,\rho^*)} (m_1|u^*_{\varepsilon}(x,s)|^p + m_2|u^*_{\varepsilon}(x,s)|^q)dx\\
&\leq 2K|B(x_0,\rho^*)| + E(0) + C(\rho^*, m_1)|\nabla u^*_{\varepsilon}(t)|^p_{2,B(x_0,\rho^*)}\\
&+ C(\rho^*, m_2)|\nabla u^*_{\varepsilon}(t)|^q_{2,B(x_0,\rho^*)} < C, \text{ if } 0 \leq t \leq \rho^2/2.
\end{aligned}$$

By integrating in time (3.19) up to $\rho^*/2$, we deduce from Alaoglu's theorem the existence of a subsequence, denoted also by u^*_{ε}, for which we have the convergences:

$$\begin{aligned}
&u^*_{\varepsilon} \to u^* \text{ weak star in } L^{\infty}(0,\rho^*/2; H^1_0(B(x_0,\rho^*)))\\
&u^*_{\varepsilon_t} \to u^*_t \text{ weak star in } L^{\infty}(0,\rho^*/2; L^2(B(x_0,\rho^*))).
\end{aligned} \qquad (3.20)$$

Also, by Aubin's theorem and the Rellich-Kondrachov compactness embedding theorem we have the convergence

$$u^*_{\varepsilon} \to u^* \text{ strongly in } L^2((0,\rho^*/2) \times B(x_0,\rho^*))$$

so for a subsequence we have

$$u^*_{\varepsilon}(x,t) \to u^*(x,t) \text{ a.e. } (x,t) \in B(x_0,\rho^*) \times (0,\rho^*/2). \qquad (3.21)$$

We will show after the construction of the Lipschitz approximations f_{ε} that this is enough to obtain $f_{\varepsilon}(x,t,u^*_{\varepsilon}(x,t)) \to f(x,t,u^*(x,t))$ in $L^1(B(x_0,\rho^*) \times (0,\rho^*/2))$, hence, also in the sense of distributions.

A monotonicity argument will be applied in order to pass to the limit in the nonlinear dissipative term $g(x,t,u^*_{\varepsilon_t})$. From (A4), the energy identity and the bounds on F_{ε} obtained above, we have:

$$\begin{aligned}
&|u^*_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho^*)} + |\nabla u^*_{\varepsilon}(t)|^2_{2,B(x_0,\rho^*)} + \int_0^t \int_{B(x_0,\rho^*)} |u^*_{\varepsilon_t}(x,s)|^{m+1} dxds\\
&\leq |u^*_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho^*)} + |\nabla u^*_{\varepsilon}(t)|^2_{2,B(x_0,\rho^*)} + \int_0^t \int_{B(x_0,\rho^*)} g(x,s,u^*_{\varepsilon_t}(x,s))\\
&\qquad \cdot u^*_{\varepsilon_t}(x,s) dxds \leq C. \qquad (3.22)
\end{aligned}$$

Therefore, we can again extract a subsequence u^*_{ε} such that:

$$\begin{aligned}
&u^*_{\varepsilon_t} \rightharpoonup u^*_t \text{ in } L^{m+1}((0,\rho^*/2) \times B(x_0,\rho^*))\\
&g(x,t,u^*_{\varepsilon_t}) \rightharpoonup \xi \text{ in } L^{(m+1)'}((0,\rho^*/2) \times B(x_0,\rho^*)).
\end{aligned} \qquad (3.23)$$

Passing to the limit in ε we obtain (we drop the * symbol for u in the sequel)

$$u_{tt} - \Delta u + f(x,t,u) + \xi = 0 \text{ in the sense of distributions.} \tag{3.24}$$

We need to verify that $\xi = g(x,t,u_t)$. By (3.20) and (3.22),

$$|u_t(t)|^2_{2,B(x_0,\rho^*)} + |\nabla u(t)|^2_{2,B(x_0,\rho^*)} + \liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} g(x,s,u_{\varepsilon t}(x,s)) u_{\varepsilon t}(x,s)\,dxds \le C. \tag{3.25}$$

By the monotonicity of g we also have that:

$$\int_0^t \int_{B(x_0,\rho^*)} (g(x,s,u_{\varepsilon t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon t}(x,s) - \phi(x,s))\,dxds \ge 0 \tag{3.26}$$

for every $\phi \in L^{m+1}((0,t)\times B(x_0,\rho^*))$. The following inequality is proven below:

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} g(x,s,u_{\varepsilon t}(x,s)) u_{\varepsilon t}(x,s)\,dxds \le \int_0^t \int_{B(x_0,\rho^*)} \xi u_t(x,s)\,dxds. \tag{3.27}$$

At the moment assume that (3.27) is valid. Then we have that

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} (g(x,s,u_{\varepsilon t}(x,s)) - g(x,s,\phi(x,s)))(u_{\varepsilon t} - \phi(x,s))\,dxds \le \int_0^t \int_{B(x_0,\rho^*)} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))\,dxds. \tag{3.28}$$

By combining (3.26) and (3.28) we obtain:

$$\int_0^t \int_{B(x_0,\rho^*)} (\xi(x,s) - g(x,s,\phi(x,s)))(u_t(x,s) - \phi(x,s))\,dxds \ge 0,$$

for all $t < \rho^*/2$, so by passing to the limit as $t \to \rho^*/2$, it holds also for $t = \rho^*/2$. We choose ϕ appropriately ($\phi_\pm := u_t \pm \lambda v$ for $\lambda > 0$) and take v arbitrary in $C_c^\infty(B(x_0,\rho^*))$. Let $\lambda \to 0$ for both choices, ϕ_+, respectively ϕ_-, to obtain the desired equality $\xi = g(u_t)$.

Remark: One can also use Lemma 1.3 page 42 in [1] to obtain $\xi = g(u_t)$ from (3.27).

Proof of inequality (3.27). We mention here that this is the only place in this work where the range of the exponents p and m has to be restricted by $p + \frac{p}{m} < 2^*$ or by $p < 2^*/2$ (except in the cases $m = 0$ or $m = 1$ when p belongs to the full subcritical interval $(1, 2^* - 1)$).

In order to obtain (3.27), it is enough to show

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} (g(x,s,u_{\varepsilon_t}(x,s)) - \xi(x,s)) \cdot (u_{\varepsilon_t}(x,s) - u_t(x,s))dxds \leq 0, \quad (3.29)$$

due to $(3.20)_2$ and $(3.23)_2$.

Note now that in order to prove (3.29) it is enough to have the similar equality for the source terms; i.e.,

$$\liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} (f_\varepsilon(x,s,u_\varepsilon(x,s)) - f(x,s,u(x,s))) \cdot (u_{\varepsilon_t}(x,s) - u_t(x,s))dxds = 0. \quad (3.30)$$

This is motivated by the following argument. By the energy identity we have

$$\begin{aligned}\int_0^t \int_{B(x_0,\rho^*)} &(f_\varepsilon(u_\varepsilon) - f(u))(u_{\varepsilon_t} - u_t)dxds \\ &+ \int_0^t \int_{B(x_0,\rho^*)} (g(u_{\varepsilon_t}) - \xi)(u_{\varepsilon_t} - u_t)dxds \\ &= - \int_0^t \int_{B(x_0,\rho^*)} \left(|u_{\varepsilon_t} - u_t|^2 + |\nabla u_\varepsilon - \nabla u|^2\right) dxds \leq 0\end{aligned}$$

where the right-hand side above is bounded below due to (3.19), and it is non-positive. We deduce that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} &(f_\varepsilon(u_\varepsilon) - f(u))(u_{\varepsilon_t} - u_t)dxds \\ &+ \liminf_{\varepsilon\to 0} \int_0^t \int_{B(x_0,\rho^*)} (g(u_{\varepsilon_t}) - \xi)(u_{\varepsilon_t} - u_t)dxds \\ &\leq \liminf_{\varepsilon\to 0} \Big(\int_0^t \int_{B(x_0,\rho^*)} (f_\varepsilon(u_\varepsilon) - f(u))(u_{\varepsilon_t} - u_t)dxds \\ &\qquad + \int_0^t \int_{B(x_0,\rho^*)} (g(u_{\varepsilon_t}) - \xi)(u_{\varepsilon_t} - u_t)dxds \Big) \leq 0,\end{aligned}$$

so if one has (3.30), then (3.29) follows.

In order to prove (3.30) we multiply out the quanitities in the integrand and show convergence for each of them. We start with the study of the "non-mixed" product $f_\varepsilon(u_\varepsilon)u_{\varepsilon_t}$. We have the equality

$$\begin{aligned}&\int_0^t\int_{B(x_0,\rho^*)} f_\varepsilon(x,s,u_\varepsilon(x,s))u_{\varepsilon_t}(x,s)dxds\\&=\int_{B(x_0,\rho^*)} F_\varepsilon(x,s,u_\varepsilon(x,s))dx|_{s=0}^{s=t}-\int_0^t\int_{B(x_0,\rho^*)} F_{\varepsilon_t}(x,s,u_\varepsilon(x,s))dxds,\end{aligned}$$

where we notice that we can pass to the limit in the first term of the right-hand side by (3.19) combined with the condition of subcritical growth for F. For the second term, by (A2), we have $|F_{\varepsilon_t}(u_\varepsilon)|\le K|u_\varepsilon|$, and since u_ε is bounded in L^1 (as a consequence of the Sobolev embedding theorem) by the Lebesgue dominated convergence we get $f_\varepsilon(x,t,u_\varepsilon)\to f(x,t,u)$ in $L^1((0,t)\times B(x_0,\rho^*))$.

The analysis of the "mixed" terms (which are $f_\varepsilon(u_\varepsilon)u_t$ and $f(u)u_{\varepsilon_t}$) will, however, impose some restrictions on the exponents p and m. We first analyze $f_\varepsilon(u_\varepsilon)u_t$ which converges almost everywhere to $f(u)u_t$ by (3.34). By Egoroff's theorem for every $\delta>0$ there exists a set $A\subset(0,t)\times B(x_0,\rho^*)$ with $|A|<\delta$ such that $f_\varepsilon(u_\varepsilon)u_t\to f(u)u_t$ uniformly (hence, in L^1) on $(0,t)\times B(x_0,\rho^*)\backslash A$. We write

$$\begin{aligned}\int_0^t\int_{B(x_0,\rho^*)} f_\varepsilon(u_\varepsilon)u_tdxds=\int_{(0,t)\times B(x_0,\rho^*)\backslash A} f_\varepsilon(u_\varepsilon)u_tdxds&\\+\int_A f_\varepsilon(u_\varepsilon)u_tdxds.&\quad(3.31)\end{aligned}$$

Due to the uniform convergence of $f_\varepsilon(u_\varepsilon)u_t\to f(u)u_t$ on $(0,t)\times B(x_0,\rho^*)\backslash A$, we have

$$\lim_{\varepsilon\to0}\int_{(0,t)\times B(x_0,\rho^*)\backslash A} f_\varepsilon(u_\varepsilon)u_tdxds=\int_{(0,t)\times B(x_0,\rho^*)\backslash A} f(u)u_tdxds.\quad(3.32)$$

In order to analyze the integral on A from (3.31) we apply Hölder's inequality with conjugate exponents α,β, and γ:

$$\int_A|f_\varepsilon(u_\varepsilon)u_t|dxds\le C\Big(\int_A|u_\varepsilon|^{\alpha p}dxds\Big)^{\frac{1}{\alpha}}\Big(\int_A|u_t|^{\beta}dxds\Big)^{\frac{1}{\beta}}|A|^{\frac{1}{\gamma}}.\quad(3.33)$$

Our goal is to bound the first two factors on the right-hand side above, and to this end we have two options for choosing α, β, and γ. First we take

$$\alpha=\frac{2^*}{p},\quad\beta=2,\quad\gamma=\frac{2\cdot2^*}{2^*-2p},$$

and by the Sobolev embedding theorem and by (3.18) we have the desired bounds in (3.33) if $\gamma > 0$. The positivity of γ amounts to $p < \frac{2^*}{2}$ which is condition (a) in (A8). The second choice is

$$\alpha = \frac{m+1}{m-(m+1)\eta}, \quad \beta = m+1, \quad \gamma = \frac{1}{\eta},$$

for some $0 < \eta < 1$. We need to impose that $\alpha p \leq 2^*$ and by letting $\eta \to 0\,(\eta \neq 0)$, we get the restriction $p + \frac{p}{m} < 2^*$ (condition (b) in (A8)).

Now we go back in (3.31) and take $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}$ on both sides. First, in (3.32) take $\lim_{\delta\to 0}$ and notice that we can bound the integrand the same way as in (3.33), and since we have the convergence of the sets $(0,t) \times B(x_0,\rho^*)\backslash A \to (0,t) \times B(x_0,\rho^*)$ as $\delta \to 0$, by the Lebesgue dominated convergence theorem we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{(0,t)\times B(x_0,\rho^*)\backslash A} f_\varepsilon(u_\varepsilon)u_t dxds = \int_{(0,t)\times B(x_0,\rho^*)} f(u)u_t dxds.$$

From (3.33) we have that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_A |f_\varepsilon(u_\varepsilon)u_t| dxds \leq \lim_{\delta\to 0} C|A|^{\frac{1}{\gamma}} \lim_{\varepsilon\to 0} M = 0,$$

where M is a bound for the first two factors on the right-hand side of (3.33). Thus we obtained:

$$\lim_{\varepsilon\to 0}\int_0^t\int_{B(x_0,\rho^*)} f_\varepsilon(u_\varepsilon)u_t dxds = \int_0^t\int_{B(x_0,\rho^*)} f(u)u_t dxds.$$

For the analysis of the second "mixed" term $f(x,t,u)u_{\varepsilon_t}$ we first integrate by parts :

$$\begin{aligned}\int_0^t\int_{B(x_0,\rho^*)} f(x,t,u)u_{\varepsilon_t} dxds &= \int_{B(x_0,\rho^*)} f(x,t,u)u_\varepsilon dxds\Big|_{s=0}^{s=t} \\ &\quad - \int_0^t\int_{B(x_0,\rho^*)} f_t(x,t,u)u_\varepsilon dxds - \int_0^t\int_{B(x_0,\rho^*)} f_u(x,t,u)u_t u_\varepsilon dxds.\end{aligned}$$

In the first two terms on the right-hand side above we use the fact that $u_\varepsilon \to u$ strongly in L^2 and (A2) to obtain convergence of the integrals with no additional restrictions. For the third term we use the same argument involving Egoroff's theorem as we did above in the proof of convergence of $f_\varepsilon(u_\varepsilon)u_t$. The analysis is similar and yields the same conditions, so we omit it.

If $m = 0$ (no damping) or $m = 1$ (Lipschitz damping) then one does not need the monotonicity argument in order to obtain $g(u_t) = \xi$, only $(3.20)_2$

and $(3.23)_1$. Since there is no other restriction imposed on p, these values for p and m cover the case (c) in (A8).

If we assume (A2)*, then (3.19):

$$|u^*_{\varepsilon_t}(t)|^2_{2,B(x_0,\rho)} + |\nabla u^*_\varepsilon(t)|^2_{2,B(x_0,\rho)} \leq C$$

holds for every $\rho > 0$ (note that we do not need Sattinger's argument, therefore no smallness conditions are imposed). The positivity of F_ε and the energy identity give us L^{p+1} bounds for the solution which will yield convergence of the source terms by Lemma 1.3 page 13 in [11] (this argument does not require Sobolev's embedding theorem, hence we could apply it for *any* p if we were able to eliminate the bound $p < 2^* - 1$ in the monotonicity argument above). We cut the initial data only so that we can work on a bounded domain where the compactness arguments allow us to pass to the limit in our sequence of approximations. The argument that follows is identical to the one used before. In the next section we will show how the local time of existence $\rho^*/2$ is replaced by any time T under hypothesis (A2)*, thus obtaining global existence.

Construction of the Lipschitz approximations $\mathbf{f}_\varepsilon$. Take η_ε a smooth cutoff function with

(1) $0 \leq \eta_\varepsilon(v) \leq 1$;
(2) $\eta_\varepsilon(v) = 1$, if $|v| < \frac{1}{\varepsilon}$;
(3) $\eta_\varepsilon(v) = 0$, if $|v| > \frac{2}{\varepsilon}$;
(4) $|\eta'_\varepsilon(v)| \leq C\varepsilon$.

Construct $f_\varepsilon(x,t,u) := f(x,t,u)\eta_\varepsilon(u)$. Then

$$f_{\varepsilon_u}(x,t,u) = \begin{cases} f_u(x,t,,u), & \text{if } |u| < \dfrac{1}{\varepsilon} \\ f_u(x,t,u)\eta_\varepsilon(u) + f(x,t,u)\eta'_\varepsilon(u), & \text{if } \dfrac{1}{\varepsilon} < |u| < \dfrac{2}{\varepsilon} \\ 0, & \text{otherwise.} \end{cases}$$

By the assumption (A0) we get that

(1) if $|u| < \frac{1}{\varepsilon}$, then $|f_{\varepsilon_u}| \leq |f_u| \leq k\left(\frac{2}{\varepsilon}\right)$;
(2) if $\frac{1}{\varepsilon} < |u| < \frac{2}{\varepsilon}$, then $|f_{\varepsilon_u}| \leq |f_u| + |f|C\varepsilon \leq k\left(\frac{2}{\varepsilon}\right) + C\varepsilon k\left(\frac{2}{\varepsilon}\right)\frac{2}{\varepsilon}$ $= (2C+1)k\left(\frac{2}{\varepsilon}\right)$;
(3) if $|u| > \frac{2}{\varepsilon}$, then $|f_{\varepsilon_u}| = 0$.

Therefore, f_ε is Lipshitz in u with the Lipschitz constant $(2C+1)k\left(\frac{2}{\varepsilon}\right)$.

We prove that if $u_\varepsilon(x,t) \to u(x,t)$ almost everywhere as $\varepsilon \to 0$ then:

$$f_\varepsilon(x,t,u_\varepsilon(x,t)) \to f(x,t,u(x,t)) \text{ a.e..}$$

We have that (we drop the x, t arguments for the functions u and u_ε):

$$\begin{aligned}|f_\varepsilon(x,t,u_\varepsilon) - f(x,t,u)| \leq& |f_\varepsilon(x,t,u_\varepsilon) - f_\varepsilon(x,t,u)| + |f_\varepsilon(x,t,u) - f(x,t,u)|\\ \leq& |f(x,t,u_\varepsilon)\eta_\varepsilon(u_\varepsilon) - f(x,t,u_\varepsilon)\eta_\varepsilon(u)| + |f(x,t,u_\varepsilon)\eta_\varepsilon(u) - f(x,t,u)\eta_\varepsilon(u)|\\ &+ |f(x,t,u)\eta_\varepsilon(u) - f(x,t,u)|\\ \leq& |f(x,t,u_\varepsilon)||\eta_\varepsilon(u_\varepsilon) - \eta_\varepsilon(u)| + |\eta_\varepsilon(u)||f(x,t,u_\varepsilon) - f(x,t,u)|\\ &+ |f(x,t,u)||\eta_\varepsilon(u) - 1|\\ \leq& |f(x,t,u_\varepsilon)| \max_v |\eta'_\varepsilon(v)||u_\varepsilon - u| + |f(x,t,u_\varepsilon) - f(x,t,u)|\\ &+ |f(x,t,u)||\eta_\varepsilon(u) - 1|.\end{aligned}$$

We conclude that

$$f_\varepsilon(x,t,u_\varepsilon(x,t)) \to f(x,t,u(x,t)) \text{ a.e.,} \tag{3.34}$$

since f and η_ε are continuous in u, $u_\varepsilon(x,t) \to u(x,t)$ almost everywhere, and $\eta_\varepsilon \to 1$ almost everywhere.

Since $f_\varepsilon(x,t,u) = \eta_\varepsilon(u)f(x,t,u)$, then from the assumption (A1) on f, we have that $|f_\varepsilon(x,t,u_\varepsilon)| \leq m_1|u_\varepsilon|^p + m_2|u_\varepsilon|^q$.

By (3.19)) and the Rellich-Kondrachov embedding theorem, since $1 < q < p < 2^* - 1$ we have that u_ε^p and u_ε^q converge strongly in L^1 (up to a subsequence), hence by the Lebesgue dominated convergence theorem $f_\varepsilon(x,t,u_\varepsilon) \to f(x,t,u)$ strongly in L^1. As an immediate consequence we have convergence in the sense of distributions for the source terms.

Thus, we proved that our approximations have the desired properties.

Step 4. We now consider the problem on the entire space $\mathbb{R}^n$. In order to eliminate the restriction of working with "small" initial data with compact support, we use a "patching" of solutions argument due to M. Crandall and L. Tartar who applied it in [19] to show global existence of a solution for the Broadwell model. This step will require us to carefully assemble all the results obtained in the previous steps.

Let (u_0, u_1) be a pair of initial data on $\mathbb{R}^n$ that satisfies the assumptions of Theorem 3.1. The recipe for constructing our solutions from general initial data is as follows:

Step 4.1. Cut the initial data in small pieces on bounded domains and for each piece obtain global existence of solutions for the approximate problems with Lipschitz source terms f_ε.

Step 4.2. For each bounded domain, obtain bounds for $|\nabla u_\varepsilon|_2$ and pass to the limit in the approximate solutions; hence, we obtain existence for the problem with a *general* source term.

Step 4.3. Up to some time $T < 1$, "patch all solutions" obtained in Step 4.2 to obtain a solution for the problem with a *general* source term with initial data on $\mathbb{R}^n$.
Step 4.4. Show that the solution defined in Step 4.3 is a well-defined function and it is the solution generated by the initial data (u_0, u_1).

Here is a detailed discussion of the above construction.
Step 4.1. Let $d > 0$. Consider a lattice of points x_k, $k \in \mathbb{N}$ in $\mathbb{R}^n$ situated at a distance d away from each other, such that in every ball of radius d we find at least one x_k. With ρ^* given by (3.15) and (3.11) (where ρ^* depends only on the norms of the initial data), construct the balls B_k of radius $\rho^*/2$ centered at x_k. The procedure outlined in Step 2 for truncating the initial data around x_0 to obtain a "small piece" denoted by (u_0^*, u_1^*), will be used now to construct around each x_k the truncations $(u_{0,k}^*, u_{1,k}^*)$ which will satisfy the "smallness" assumptions

$$|\nabla u_{0,k}^*|_{B_k} < \alpha, \quad \frac{1}{2}|u_{1,k}^*|_{B_k}^2 + \frac{1}{2}|\nabla u_{0,k}^*|_{B_k}^2 + \int_{B_k} F(x, 0, u_{0,k}^*(x))\, dx + K|B_k| < \Phi(\alpha).$$

On each of the balls B_k we apply Theorem 2.3 to obtain global existence of solutions $u_{\varepsilon,k}^*$ for the problem (SWB) with initial data $(u_{0,k}^*, u_{1,k}^*)$ and with the Lipschitz approximations f_ε for the source term (see the construction f_ε at the end of Step 3).
Step 4.2. At this point, the arguments of Step 3 for passing to the limit as $\varepsilon \to 0$ in the sequence of approximate problems are applied, where x_0 is successively replaced by x_k. First, we apply Sattinger's argument to estimate $|\nabla u_\varepsilon|_2$ on each of the balls $B(x_k, \rho^*/2)$. (Note that we need to make use of the smallness assumptions written in Step 4.1.) The convergence $u_{\varepsilon,k}^* \to u_k^*$ takes place on every domain $B_k \times (0, \rho^*/2)$, so we obtain a global solution to the boundary-value problem (SWB) for $\Omega = B_k$, for every k.
Step 4.3. The solutions u_k^* found in Step 4.2 will now be "patched" together to obtain our general solution. First, we need to introduce the following notation. For $k \in \mathbb{N}$, let $C_k := \{(y, s) \in \mathbb{R}^3 \times [0, \infty); |y - x_k| \le \rho^*/2 - s\}$ be the backward cones which have their vertices at $(x_k, \rho^*/2)$. For d small enough (i.e., for $0 < d < \rho^*/2$) any two neighboring cones C_k and C_j will intersect. For every set of intersection $I_{k,j} := C_k \cap C_j$ the maximum value for time contained in it is equal to $(\rho^* - d)/2$ (see Figure 3 below).

For $t < \rho^*/2$ we define the piecewise function:

$$u(x,t) := u_k^*(x,t), \text{ if } (x,t) \in C_k. \tag{3.35}$$

This solution is defined only up to time $(\rho^* - d)/2$, since the cones do not cover the entire strip $\mathbb{R}^n \times (0, \rho^*/2)$. By letting $d \to 0$ we can obtain a solution well defined up to time $\rho^*/2$. Thus, we have u defined up to time $\rho^*/2$, which is the height of all cones C_k. Every pair $(x,t) \in \mathbb{R}^n \times (0, \rho^*/2)$ belongs to at least one C_k, so in order to show that this function from (3.35) is well defined, we need to check that it is single valued on the intersection of two cones. Also, we need to show that the above function is the solution generated by the pair of initial data (u_0, u_1). Both proofs will be done in the next step.

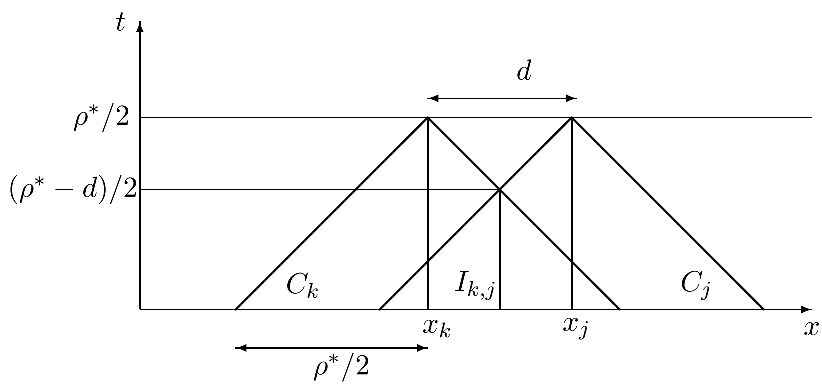


Figure 3: The intersection of the cones C_k and C_j

Step 4.4. In order to prove the properties that we set out to do in this step, we will go back and look at the solutions u_k^* as limits of the approximation solutions $u_{k,\varepsilon}^*$.

For each $k \in \mathbb{N}$ we have $(u_{0,k}^*, u_{1,k}^*) = (u_0, u_1)$ for every $x \in B_k = \{y \in \mathbb{R}^n : |y - x_k| < \rho^*/2\}$ (see the construction of the truncations $(u_{0,k}^*, u_{1,k}^*)$ in Step 2). Therefore, $u_{\varepsilon,k}^*$ (defined in Step 4.1) is an approximation of the solution generated by the initial data (u_0, u_1) on C_k (from the uniqueness property given by Part 2 of Proposition 2.4). We let $\varepsilon \to 0$ (use the argument from Step 3) to show that the solution u on each C_k is generated by the initial data (u_0, u_1).

To show that u defined by (3.35) is a proper function, we use the same result of uniqueness given by the finite speed of propagation. First note that for $n \geq 3$ the intersection $I_{k,j}$ is not a cone, but it is contained by the cone $C_{k,j}$ with the vertex at $((x_k + x_j)/2, (\rho^* - d)/2)$ of height $(\rho^* - d)/2$. In this cone we use the uniqueness asserted by the finite speed of propagation as follows. First note that the cones $C_{k,j}$ contain the sets $I_{k,j}$, but $C_{k,j} \subset C_k \cup C_j$. In C_k and C_j we have the two solutions $u_{k,\varepsilon}^*$ and $u_{j,\varepsilon}^*$ (see construction in 4.1); hence, in $C_{k,j}$ we now have defined two solutions. Since $u^*_{k,\varepsilon}$ and $u^*_{j,\varepsilon}$ start with the same initial data $((u^*_{0,k}, u^*_{1,k}) = (u_0, u_1) = (u^*_{0,j}, u^*_{1,j})$ on $B_k \cap B_j)$, they are equal. We let $\varepsilon \to 0$ to obtain $u^*_k = u^*_j$ in $C_{k,j}$, and since $I_{k,j} \subset C_{k,j}$ we proved $u^*_k = u^*_j$ on $I_{k,j}$. Therefore, u is a single-valued (proper) function.

Finally, the fact that this constructed function u is a solution to the Cauchy problem (SW) is immediate since it satisfies both the wave equation and the initial conditions.

The above method of using cutoff functions and "patching" solutions based on uniqueness will work the same way in the case when we additionally assume (A2)*. Since we can choose the height of the cones as large as we wish, the solutions exist globally in time under the positivity hypothesis for F. $\square$

Remark 1. This proof works in the variable coefficient case, i.e., for the equation

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x,t))_{x_j} + f(x,t,u) + g(x,t,u_t) = 0, \tag{3.36}$$

where to the assumptions (A0)-(A7), we add the following assumptions concerning the coefficients a_{ij}. For every $1 \le i,j \le n$, we impose that a_{ij} are

(1) bounded: $a_{ij} \in L^\infty(\mathbb{R}^n)$;
(2) symmetric: $a_{ij} = a_{ji}$;
(3) elliptic: $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge k|\xi|^2$, $k > 0$, for every $\xi \in \mathbb{R}^n$ with components ξ_i.

This generalization is mainly possible due to the fact that the arguments used in the proof of Theorem 2.1 and in the finite propagation speed property do not critically rely on the fact that the coefficients are constant. The rest of the proof can be easily adjusted.

Remark 2. The local existence result obtained under the assumption (A1)(a) can not be extended to a global existence theorem, as blow-up results in the non-coercive case show that the solution may go to infinity in L^∞ norm in finite time (see [5], [7]). More precisely, in [7] it was shown that if $1 < m < p < \frac{n}{n-2}$, the solution of the equation:

$$u_{tt} - \Delta u - u|u|^{p-1} + u_t|u_t|^{m-1} = 0$$

will exist only locally in time if its initial data has sufficiently large negative energy.

Remark 3. The assumption (A0) for f can be relaxed in the sense that we only need a function k such that

$$\text{if } |x|, t, |u| \leq r, \text{ then } |f_u(x,t,u)| \leq k(r).$$

Remark 4. In the case of (A2)* the bound $p < 2^* - 1$ can be replaced by the less restrictive $p + \frac{p}{m} < 2^*$ (see the discussion following the monotonicity argument for (A2)*).

Remark 5. In the growth assumption (A1)(ii) we can allow $2 \leq q < p$, instead of $2 < q < p$, if $m_2 < C^*$, where C^* is the constant from Sobolev's inequality, so the potential well function Φ will have a quadratic term with a positive coefficient in front.

4. APPENDIX

This section is dedicated to obtaining global existence of weak solutions for semilinear wave equations with Lipschitz source terms and monotone damping. Such a result can be obtained via semigroup theory (see for example [3]), but in order to make the results of this paper self contained, we include a proof based solely on estimates. The ideas of this proof can be found in the classical works of V. Barbu [1], and J.-L. Lions [12].

At first, we present a lemma that collects a series of properties of the Yosida approximation. Given a function $g(x,t,v)$ that satisfies (A3)-(A7) we define the Yosida approximation of g *in the third argument*, v, by :

$$g^\lambda(x,t,v + \lambda g(x,t,v)) = g(x,t,v). \tag{4.1}$$

With the aid of the function

$$H_{\lambda,x,t}(v) = v + \lambda g(x,t,v)$$

we can write

$$g^\lambda(x,t,v) = g(x,t,H^{-1}_{\lambda,x,t}(v)), \tag{4.2}$$

where H^{-1} denotes the inverse of H. We denote by:

$$G^\lambda(x,t,v) = \int_0^v g^\lambda(x,t,y)dy.$$

Then, we have the following:

Lemma 4.1. (Properties of the Yosida approximation) *For $g(x,t,v)$, a function which is increasing and differentiable in v, let g^λ be given as in (4.2). Then, the following hold*

(i) $(A4)_\lambda$: g^λ *is increasing;*

(ii) g^λ *is a Lipschitz function in* v *of constant* $\frac{1}{\lambda}$*; i.e.,*

$$|g^\lambda(x,t,v_1) - g^\lambda(x,t,v_2)| \le \frac{1}{\lambda}|v_1 - v_2|;$$

(iii) $\lambda g^\lambda(x,t,v) = v - H^{-1}_{\lambda,x,t}(v)$*;*

(iv) $G^\lambda \ge 0$*;*

(v) $(A6)$ *implies*

$$(A6)_\lambda : \quad |\nabla_x g^\lambda(x,t,v)| \le C|v|;$$

(vi) $(A7)$ *implies*

$$(A7)_\lambda : \quad |G^\lambda_t(x,t,v)| \le C|v|^2;$$

(vii) $G^\lambda(x,t,v) \le G(x,t,v)$*, for every* x,t,v*; hence*

$$||G^\lambda(t,v)||_{L^1(\Omega)} \le ||G(t,v)||_{L^1(\Omega)}.$$

Proof. In the equations below the arguments x and t will be suppressed whenever they do not play a significant role.

(i) First note that H^{-1} is an increasing function, being the inverse of an increasing function. By (4.2) g^λ is a composition of increasing functions, therefore it inherits the same monotonicity.

(ii) We differentiate with respect to v the equality (4.1) and obtain:

$$g^\lambda_v(v + \lambda g(v)) = \frac{g_v(v)}{1 + \lambda g_v(v)}. \tag{4.3}$$

Since $g_v(v) \ge 0$ for every v, we get $0 \le g^\lambda_v \le \frac{1}{\lambda}$.

The fact that $0 \le g^\lambda_v \le \frac{1}{\lambda}$ implies:

$$|g^\lambda(x,t,v_1) - g^\lambda(x,t,v_2)| \le \int_{v_1}^{v_2} |g^\lambda_v(x,t,y)|dy \le \frac{1}{\lambda}|v_1 - v_2|.$$

(iii) It is enough to show that $H^{-1}_{\lambda,x,t} = (I + \lambda g)^{-1} = I - \lambda g^\lambda$, where $I : \mathbb{R} \to \mathbb{R}$ is the identity function; i.e. $I(v) = v$ and the inverse functions are taken only with respect to the v argument. This equality is true, since $(I - \lambda g^\lambda)(I + \lambda g) = I$ is the same as $g = g^\lambda(I + \lambda g)$, which is equivalent to (4.2).

(iv) We have $g^\lambda(x,t,0) = 0$ (by the definition of g^λ and by $g(x,t,0) = 0$). Therefore, by (i) $g^\lambda(v) \ge 0$, if $v \ge 0$ and $g^\lambda(v) < 0$, if $v < 0$, and this implies $G^\lambda \ge 0$.

(v) For simplicity, denote by $v^\lambda(x,t,v) := (I + \lambda g(x,t))^{-1}(v)$, so that

$$v^\lambda(x,t,v) + \lambda g(x,t,v^\lambda) = v. \tag{4.4}$$

Then, by the definition of g^λ

$$g^\lambda(x,t,v) = g(x,t,v^\lambda). \tag{4.5}$$

We differentiate (4.4) with respect to x and obtain

$$\nabla_x v^\lambda + \lambda \nabla_x g(v^\lambda) + \lambda g_v(v^\lambda)\nabla_x v^\lambda = 0.$$

So

$$\nabla_x v^\lambda (1 + \lambda g_v(v^\lambda)) = -\lambda \nabla_x g(v^\lambda).$$

Hence, by (4.5), (A6) and $\lambda, g_v \geq 0$ we have:

$$\begin{aligned} |\nabla_x g^\lambda(v)| \leq |\nabla_x g(v^\lambda)| + |g_v(v^\lambda)\nabla_x v^\lambda| &\overset{\text{by } (A6)}{\leq} C|v^\lambda| + \frac{\lambda g_v(v^\lambda)|\nabla_x g(v^\lambda)|}{1+\lambda g_v(v^\lambda)} \\ &\leq C|v^\lambda| + |\nabla_x g(v^\lambda)| \leq C|v^\lambda|. \end{aligned}$$

The facts that g is increasing and $g(x,t,0) = 0$ imply that $v^\lambda g(v^\lambda) \geq 0$, so by squaring (4.4), we obtain $|v^\lambda| \leq |v|$, which together with the previous inequality and the hypothesis conclude the proof.

(vi) As in the previous case, we prove that $|g_t^\lambda(x,t,v)| \leq C|v|$. By integrating with respect to v, we obtain $(\mathrm{A7})_\lambda$.

(vii) In the notation of (v), since g^λ is increasing and $g^\lambda(0) = 0$, we have that if $v \leq 0$ then $g^\lambda(v) = g(v^\lambda) \leq 0$. This implies that $v^\lambda \leq 0$ since g is increasing with $g(0) = 0$. Hence v and v^λ have the same sign (the case $v \geq 0$ can be treated in an analogous way). Recall that $|v^\lambda| \leq |v|$. In the case $0 \leq v^\lambda \leq v$, by the monotonicity of g, we have that $0 \leq g(v^\lambda) = g^\lambda(v) \leq g(v)$. By integration with respect to v we obtain $G^\lambda(x,t,v) \leq G(x,t,v)$, for $v \geq 0$. In an analogous way we treat the case $v \leq v^\lambda \leq 0$. By integration with respect to the x variable, we obtain the desired inequality of the L^1 norms. $\square$

Remark. The above properties hold for g only continuous and increasing. Approximate such a function with differentiable functions (by taking convolutions with mollifiers) for which the above statements are true. Pass to the limit to obtain the same conclusions for g.

Next we present the proof of Theorem 2.1.

Proof. Existence: Under the assumptions of Theorem 2.1 consider the approximate problem:

$$\begin{cases} u_{tt}^\lambda - \Delta u^\lambda + f(x,t,u^\lambda) + g^\lambda(x,t,u_t^\lambda) = 0 \text{ in } \Omega \times (0,T); \\ (u^\lambda, u_t^\lambda)|_{t=0} = (u_0, u_1); \\ u^\lambda = 0 \text{ on } \partial\Omega \times (0,T), \end{cases} \tag{SWB$_\lambda$}$$

where g^λ is the Yosida approximation of g defined for $\lambda > 0$ by (4.1).

The approximate problem (SWB_λ) can be seen as a Lipschitz perturbation of a linear semigroup, so it has a unique weak solution for sufficiently regular initial data (see [1]). For this problem, the regularity of the initial data is propagated in time. Since for the following estimates we need sufficiently regular solutions, we approximate the initial data by C_0^∞ functions $u_{0_\varepsilon}, u_{1_\varepsilon}$. Then we pass to the limit as $\varepsilon \to 0$ in the estimates obtained, and this yields the estimates for the problem with initial data $u_0, u_1 \in H_0^1(\Omega)$, $u_0 \in H^2(\Omega)$. This argument allows us to use the multipliers below and establishes the validity of the following computations, where we drop the subscript ε.

A priori estimates. The following estimates are needed for the proof:

$$|u_t^\lambda(t)|^2_{L^2(\Omega)} + ||u^\lambda(t)||^2_{H_0^1(\Omega)} \leq C; \tag{4.6}$$

$$||u_t^\lambda(t)||^2_{H_0^1(\Omega)} + |\Delta u^\lambda(t)|^2_{L^2(\Omega)} \leq C; \tag{4.7}$$

$$\int_0^T |u_{tt}^\lambda(t)|^2_{L^2(\Omega)} dt \leq C; \tag{4.8}$$

$$\int_0^T |g^\lambda(t, u_t^\lambda)|^2_{L^2(\Omega)}\, dt \leq C, \tag{4.9}$$

for all $t \in [0, T]$, and where C is a generic constant, independent of λ.

These estimates are obtained by multiplying the equation (SWB_λ) by appropriate quantities. In order to obtain (4.6) we use the multiplier u_t^λ, integrate over the space Ω and obtain:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|u_t^\lambda(x,t)|^2 + |\nabla u^\lambda(x,t)|^2\Big)dx \leq -\int_\Omega f(x,t,u^\lambda)u_t^\lambda(x,t)dx,$$

by the monotonicity of g^λ. Integration in t and the Lipschitz assumptions on f yield:

$$\begin{aligned}
&\int_\Omega \Big(|u_t^\lambda(x,t)|^2 + |\nabla u^\lambda(x,t)|^2\Big)dx \\
&\quad\leq \int_\Omega \Big(u_1^2(x) + |\nabla u_0(x)|^2\Big)dx + 2L\int_0^t\int_\Omega |u^\lambda(x,s)||u_t^\lambda(x,s)|dxds \\
&\quad\leq \int_\Omega \Big(u_1^2(x) + |\nabla u_0(x)|^2\Big)dx + L\int_0^t\int_\Omega \Big(|u^\lambda(x,s)|^2 + |u_t^\lambda(x,s)|^2\Big)dx\,ds,
\end{aligned}$$

which by Poincaré's inequality is

$$\leq \int_\Omega \Big(u_1^2(x) + |\nabla u_0(x)|^2\Big)dx + L\int_0^t\int_\Omega \Big(C|\nabla u^\lambda(x,s)|^2 + |u_t^\lambda(x,s)|^2\Big)dx\,ds.$$

These inequalities hold for any $t \in (0,T)$, so by Gronwall we get:

$$|u_t^\lambda(t)|^2_{L^2(\Omega)} + ||u^\lambda(t)||^2_{H_0^1(\Omega)} \leq e^{CT} \int_\Omega \left(u_1^2(x) + |\nabla u_0(x)|^2\right) dx = \text{Const.}$$

Estimate (4.7) is obtained by multiplying the equation by $-\Delta u_t^\lambda$ and integrating in x. We omit the x, t arguments for the function u to write:

$$\int_\Omega \left[-u_{tt}^\lambda \Delta u_t^\lambda + \Delta u^\lambda \Delta u_t^\lambda - f(x,t,u^\lambda)\Delta u_t^\lambda \ - g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda\right] dx = 0. \tag{4.10}$$

We have

$$\int_\Omega g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda dx \leq -\int_\Omega \nabla_x g^\lambda(x,t,u_t^\lambda) \cdot \nabla u_t^\lambda dx. \tag{4.11}$$

To show (4.11), we mollify g^λ, so that its approximations are increasing and differentiable. For the approximations (denoted still g^λ) we use Green's formula where all the boundary terms are zero to write:

$$\begin{aligned}\int_\Omega g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda dx = &-\int_\Omega g_v^\lambda(x,t,u_t^\lambda)|\nabla u_t^\lambda|^2 dx \\ &-\int_\Omega \nabla_x g^\lambda(x,t,u_t^\lambda)\cdot \nabla u_t^\lambda dx \leq -\int_\Omega \nabla_x g^\lambda(x,t,u_t^\lambda)\cdot\nabla u_t^\lambda dx.\end{aligned}$$

By passing to the limit in the sequence of approximations, we have (4.11) for g^λ.

Therefore, by $(A6)_\lambda$ (the consequence of (A6) for g^λ) and by (4.11) we get:

$$\int_\Omega g^\lambda(x,t,u_t^\lambda)\Delta u_t^\lambda dx \leq C\int_\Omega \Big(|\nabla u_t^\lambda(x,t)|^2 + |u_t^\lambda(x,t)|^2\Big)dx,$$

hence, by (4.10) and the above inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|\nabla u_t^\lambda|^2 + |\Delta u^\lambda|^2\Big)dx \leq &\int_\Omega f(x,t,u^\lambda)\Delta u_t^\lambda dx \\ &+ C\int_\Omega \Big(|\nabla u_t^\lambda|^2 + |u_t^\lambda|^2\Big)dx.\end{aligned}$$

We integrate in time to obtain

$$\begin{aligned}\int_\Omega \Big(|\nabla u_t^\lambda(x,t)|^2 + |\Delta u^\lambda(x,t)|^2\Big)dx \leq &\int_\Omega \Big(|\nabla u_1(x)|^2 + |\Delta u_0(x)|^2\Big)dx \\ + \int_0^t\int_\Omega \Big(2f(x,t,u^\lambda)\Delta u_t^\lambda(x,s) &+ C|\nabla u_t^\lambda(x,s)|^2 + C|u_t^\lambda(x,s)|^2\Big)dxds.\end{aligned}$$

With (4.6) we bound $\int_0^t \int_\Omega |u_t^\lambda(x,s)|^2 dxds$. In the term that contains f, we integrate by parts with respect to t, so the above inequality becomes:

$$\int_\Omega \Big(|\nabla u_t^\lambda(x,t)|^2 + |\Delta u^\lambda(x,t)|^2\Big)dx$$
$$\leq C + C\int_0^t \int_\Omega |\nabla u_t^\lambda(x,s)|^2 dxds + 2\int_\Omega f(x,s,u^\lambda)\Delta u^\lambda(x,s)dx|_{s=0}^{s=t}$$
$$- 2\int_0^t \int_\Omega [f_t(x,t,u^\lambda) + f_u(x,t,u^\lambda)u_t^\lambda(x,s)]\Delta u^\lambda(x,s)dxds,$$

and since $|f_u| < L$, $|f_t| < C$, with the help of Young's inequality we obtain:

$$\int_\Omega \Big(|\nabla u_t^\lambda(x,t)|^2 + |\Delta u^\lambda(x,t)|^2\Big)dx \leq C + C\int_0^t \int_\Omega |\nabla u_t^\lambda(x,s)|^2 dxds$$
$$+ \frac{L}{\alpha}\int_\Omega |u^\lambda(x,t)|^2 dx + L\alpha\int_\Omega |\Delta u^\lambda(x,t)|^2 dx$$
$$+ \int_0^t \int_\Omega \Big((L + C|u_t^\lambda(x,s)|)^2 + |\Delta u^\lambda(x,s)|^2\Big)dxds,$$

which with the right choice for α, the aid of Poincaré's inequality, and by using the estimate (4.6) yields a Gronwall type inequality. This Gronwall-type inequality implies (4.7).

Remark: In the above estimate we actually used that fact that f is differentiable almost everywhere with respect to u, which is a consequence of f being Lipschitz in u.

The third estimate in our list (4.8) is obtained with the aid of the multiplier u_{tt}^λ, and by integrating in space and time. Hence,

$$\int_0^T \int_\Omega |u_{tt}^\lambda(x,s)|^2 dxds + \int_\Omega G^\lambda(x,T,u_t^\lambda(x,T))dx$$
$$= \int_\Omega G^\lambda(x,0,u_1(x))dx + \int_0^T \int_\Omega G_t^\lambda(x,s,u_t(x,s))dxds$$
$$+ \int_0^T \int_\Omega \Delta u^\lambda(x,s)u_{tt}^\lambda(x,s)dxds - \int_0^T \int_\Omega f(x,t,u^\lambda(x,s))u_{tt}^\lambda(x,s)dxds$$
$$\overset{\text{by } (A7)_\lambda}{\leq} \int_\Omega G^\lambda(x,0,u_1(x))dx + C\int_0^T |u_t(s)|^2_{L^2(\Omega)}ds + \frac{1}{2\varepsilon}\int_0^T |\Delta u^\lambda(s)|^2_{L^2(\Omega)}ds$$
$$+ \frac{\varepsilon}{2}\int_0^T |u_{tt}^\lambda(s)|^2_{L^2(\Omega)}ds + \frac{L}{2\eta}\int_0^T |u^\lambda(s)|^2_{L^2(\Omega)}ds + \frac{L\eta}{2}\int_0^T |u_{tt}^\lambda(s)|^2_{L^2(\Omega)}ds,$$

where we made use of Young's inequality with coefficients $\varepsilon, \frac{1}{\varepsilon}, \eta, \frac{1}{\eta}$. By choosing ε and η small enough, Poincaré's inequality combined with the bounds from (4.6) and (4.7) yields :

$$C\int_0^T |u_{tt}^\lambda(s)|^2_{L^2(\Omega)}ds + \int_\Omega G^\lambda(x,T,u_t^\lambda(x,T))dx \le \int_\Omega G^\lambda(x,0,u_1(x))dx + C.$$

G^λ is a positive function by Lemma 4.1_3 and $||G^\lambda(0,u_1)||_{L^1(\Omega)} \le C$ by the hypothesis and Lemma 4.1_4. These facts will imply (4.8).

We follow the same kind of argument for the last estimate in (4.9), multiplying by $g^\lambda(x,t,u_t^\lambda)$ and integrating over $(0,T)\times\Omega$.

$$\begin{aligned}\int_0^T\int_\Omega |g^\lambda(x,s,u_t^\lambda)|^2dxds &= \int_0^T\int_\Omega \Big(\Delta u^\lambda(x,s)g^\lambda(x,s,u_t^\lambda)\\ &\quad -u_{tt}^\lambda(x,s)g^\lambda(x,s,u_t^\lambda) - f(x,s,u^\lambda)g^\lambda(x,s,u_t^\lambda)\Big)\,dxds\\ &\le \int_0^T\int_\Omega\left(\frac{1}{2\varepsilon}|\Delta u^\lambda(x,s)|^2 + \frac{\varepsilon}{2}|g^\lambda(x,s,u_t^\lambda)|^2 + \frac{1}{2\eta}|u_{tt}^\lambda(x,s)|^2\right.\\ &\quad\left. +\frac{\eta}{2}|g^\lambda(x,s,u_t^\lambda)|^2 + \frac{L}{2\zeta}|u^\lambda(x,s)|^2 + \frac{\zeta}{2}|g^\lambda(x,s,u_t^\lambda)|^2\right)dxds.\end{aligned}$$

Again, choose ε,η,ζ small enough in Young's inequality and use (4.6),(4.7) and (4.8) to obtain (4.9). At this point, the estimates hold for regular solutions u_ε^λ, where we omitted the subscript ε. We let $\varepsilon\to 0$, so (4.6-4.9) take place for solutions u^λ.

Next, we will show that $(u^\lambda)_{\lambda\ge 0}$ is a Cauchy sequence in $H_0^1(\Omega)$ and $(u_t^\lambda)_{\lambda\ge 0}$ is Cauchy in $L^2(\Omega)$. We subtract the equation (SWB_μ) from (SWB_λ), multiply the result by the difference $u_t^\lambda - u_t^\mu$, and integrate over Ω to obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_t^\lambda(x,t)-u_t^\mu(x,t))^2 + |\nabla u^\lambda(x,t) - \nabla u^\mu(x,t)|^2dx\right)\\ &\quad+\int_\Omega (f(x,t,u^\lambda)-f(x,t,u^\mu))(u_t^\lambda(x,t)-u_t^\mu(x,t))dx\\ &\quad+\int_\Omega (g^\lambda(x,t,u_t^\lambda)-g^\mu(x,t,u_t^\mu))(u_t^\lambda(x,t)-u_t^\mu(x,t))dx = 0.\end{aligned}\tag{4.12}$$

By Lemma 4.1_2 we have the identity

$$u_t^\lambda = \lambda g^\lambda(u_t^\lambda) + (1+\lambda g)^{-1}(u_t^\lambda),$$

and a similar relation for u^μ. We drop the x,t arguments to write:

$$(g^\lambda(u_t^\lambda) - g^\mu(u_t^\mu))(u_t^\lambda - u_t^\mu) = (g^\lambda(u_t^\lambda) - g^\mu(u_t^\mu))(\lambda g^\lambda(u_t^\lambda) - \mu g^\mu(u_t^\mu)$$

$$+ (I+\lambda g)^{-1}(u_t^\lambda) - (I+\mu g)^{-1}(u_t^\mu)). \quad (4.13)$$

We denote $\zeta := (I+\lambda g)^{-1}(u_t^\lambda)$, $\eta := (I+\mu g)^{-1}(u_t^\mu)$, and use the definition of the Yosida approximations to get:

$$g^\lambda(u_t^\lambda) = g(I+\lambda g)^{-1}(u_t^\lambda) = g(\zeta), \quad g^\mu(u_t^\mu) = g(I+\mu g)^{-1}(u_t^\mu) = g(\eta).$$

We employ the above relations and the monotonicity of g in (4.13):

$$\begin{aligned}(g^\lambda(u_t^\lambda) - g^\mu(u_t^\mu))(u_t^\lambda - u_t^\mu) &= (g^\lambda(u_t^\lambda) - g^\mu(u_t^\mu))(\lambda g^\lambda(u_t^\lambda) - \mu g^\mu(u_t^\mu)) \\ &+ (g(\zeta)-g(\eta))(\zeta-\eta) \geq (g^\lambda(u_t^\lambda) - g^\mu(u_t^\mu))(\lambda g^\lambda(u_t^\lambda) - \mu g^\mu(u_t^\mu)).\end{aligned}$$

We integrate (4.12) with respect to time and use the Lipschitz assumption on f to arrive at the following inequalities:

$$\begin{aligned}&|u_t^\lambda(t) - u_t^\mu(t)|^2_{L^2(\Omega)} + ||u^\lambda(t) - u^\mu(t)||^2_{H_0^1(\Omega)} \\ &\leq 2L\int_0^t\int_\Omega |u^\lambda(x,s) - u^\mu(x,s)|\cdot|u_t^\lambda(x,s)u_t^\mu(x,s)|dxds \\ &\quad - 2\int_0^t\int_\Omega (g^\lambda(x,s,u_t^\lambda) - g^\mu(x,s,u_t^\mu))(\lambda g^\lambda(x,s,u_t^\lambda) - \mu g^\mu(x,s,u_t^\mu))dxds \\ &\leq L\int_0^t\int_\Omega \left(|u^\lambda(x,s) - u^\mu(x,s)|^2 + |u_t^\lambda(x,s) - u_t^\mu(x,s)|^2\right) dxds + C|\lambda-\mu| \\ &\leq C\int_0^t \left(|u_t^\lambda(s) - u_t^\mu(s)|^2_{L^2(\Omega)} + ||u^\lambda(s) - u^\mu(s)||^2_{H_0^1(\Omega)}\right) ds + C|\lambda-\mu|,\end{aligned}$$

where Poincaré's inequality was used to obtain the last inequality. An application of Gronwall's inequality shows that our sequence is Cauchy. Further explanation is due in the above argument where we used that

$$-\int_0^t\int_\Omega (g^\lambda(x,s,u_t^\lambda) - g^\mu(x,s,u_t^\mu))(\lambda g^\lambda(x,s,u_t^\lambda) - \mu g^\mu(x,s,u_t^\mu))dxds \\ \leq C(\lambda-\mu).$$

For simplicity, let us denote by a and b the following quantities: $a := g^\lambda(x,s,u_t^\lambda)$, $b := g^\mu(x,s,u_t^\mu)$ Then, it will be enough to show that:

$$-(\lambda a - \mu b)(a-b) \leq C(\lambda-\mu) \quad (4.14)$$

for some C, which can be positive or negative. There are two cases: either $\lambda = \mu$, which is trivial, or $\lambda \neq \mu$. In this second case, we can choose $C = \frac{b^2-a^2}{2}$, so the following inequalities hold: $a^2 - ab + C \geq 0$, $b^2 - ab - C \geq 0$, which implies

$$-\lambda(a^2 - ab + C) - \mu(b^2 - ab - C) \leq 0,$$

which is (4.14) rearranged. The argument is finished as we observe that $\int_0^T \int_\Omega a^2 dxds$, respectively $\int_0^T \int_\Omega b^2 dxds$, are finite due to (4.9).

Thus, we have the following convergences for the Cauchy sequences:

$$\begin{aligned} &u^\lambda(t) \to u(t) \text{ uniformly on } [0,T] \text{ in } H_0^1(\Omega)\\ &u_t^\lambda(t) \to u_t(t) \text{ uniformly on } [0,T] \text{ in } L^2(\Omega). \end{aligned} \tag{4.15}$$

In order to conclude the proof of existence (and regularity) of the solution, we remark that also the following convergences take place for a subsequence (not relabeled) of u^λ:

$u_{tt}^\lambda \to u_{tt}$ weakly in $L^2(0,T;L^2(\Omega))$ due to (4.8);

$\Delta u^\lambda \to \Delta u$ weakly in $L^2(0,T;L^2(\Omega))$ due to (4.7);

$f(x,t,u^\lambda) \to f(x,t,u)$ in $L^2(0,T;L^2(\Omega))$ due to the Lipschitz assumptions;

$g^\lambda(x,t,u_t^\lambda) \to \xi$ weakly in $L^2(0,T;L^2(\Omega))$ for some $\xi \in L^2(0,T;L^2(\Omega))$, due to (4.9).

Finally, $(4.15)_2$ implies $\xi = g(u_t)$ in $L^2(0,T;L^2(\Omega))$.

Uniqueness. Suppose that u and v are two solutions of (SWB), then the difference $u-v$ satisfies the equation:

$$(u-v)_{tt} - \Delta(u-v) + f(x,t,u) - f(x,t,v) + g(x,t,u_t) - g(x,t,v_t) = 0,$$

with initial and boundary data identically zero. As usual, we multiply the equation by $(u-v)_t$, which is allowed due to the regularity obtained above, and integrate in space and time. Therefore:

$$\begin{aligned} \int_\Omega [(u_t - v_t)(t,x)]^2 + |\nabla(u-v)(t,x)|^2 dx \leq -\int_0^t \int_\Omega (f(x,t,u) - f(x,t,v))\\ \cdot (u_t(x,s) - v_t(x,s)) + (g(x,s,u_t) - g(x,s,v_t))(u_t(x,s) - v_t(x,s)) dsdx. \end{aligned}$$

The same ingredients that we used before, the Lipschitz assumptions on f, the monotonicity of g, and the Cauchy and Gronwall inequalities give us $u - v = 0$. Thus, the solutions of (SWB) are unique. $\square$

Acknowledgements. I wish to express my gratitude to my thesis advisor, Luc Tartar, for his advice and patience while working on this result. I would like to thank Mohammad Rammaha for his careful reading of the manuscript. My most sincere gratitude goes to Irena Lasiecka whose valuable comments led to a significant improvement of this paper.

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